WEIL-PETERSSON METRIC ON THE UNIVERSAL TEICHMÜLLER SPACE I: CURVATURE PROPERTIES AND CHERN FORMS

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Abstract. We prove that the universal Teichmüller space $T(1)$ carries a new structure of a complex Hilbert manifold. We define a Weil-Petersson metric on $T(1)$ by Hilbert space inner products on tangent spaces, compute its Riemann curvature tensor, and show that $T(1)$ is a Kähler -Einstein manifold with negative Ricci and sectional curvatures. We introduce and compute Mumford-Miller-Morita characteristic forms for the vertical tangent bundle of the universal Teichmüller curve fibration over the universal Teichmüller space. As an application, we derive Wolpert curvature formulas for the finite-dimensional Teichmüller spaces from the formulas for the universal Teichmüller space.

Contents

1. Introduction

The universal Teichmüller space $T(1)$ is the simplest Teichmüller space that bridges spaces of univalent functions and general Teichmüller spaces. Introduced by Bers [Ber65, Ber72, Ber73], the universal Teichmüller space is an infinite-dimensional complex manifold modeled on a Banach space; it contains Teichmüller spaces of Riemann surfaces as complex submanifolds. The universal Teichmüller space $T(1)$ also came to the forefront with the advent of the string theory. It contains as a complex submanifold an infinite-diimensional complex Fréchet manifold $\text{Möb}(S^1)\backslash\text{Diff}_+(S^1)$, which plays an important role in one of the approaches to non-perturbative bosonic closed string field theory based on Kähler geometry [BR87a, BR87b]. The manifold $\text{Möb}(S^1)\backslash\text{Diff}_+(S^1)$ — a homogeneous space of the Lie group $\text{Diff}_+(S^1)$, also has an interpretation as a coadjoint orbit of the Bott-Virasoro group, and as such carries a natural right-invariant Kähler metric [Kir87, KY87].

The complex geometry of the finite-dimensional Teichmüller spaces — Teichmüller spaces $T(\Gamma)$ of cofinite Fuchsian groups, has been extensively studied in the context of Ahlfors-Bers deformation theory of complex structures on Riemann surfaces. In particular, A. Weil defined a natural Hermitian metric on $T(\Gamma)$ by the Petersson inner product on the tangent spaces. Called Weil-Petersson metric, it was shown to be a Kähler metric by Weil and Ahlfors. In his seminal paper [Ahl62] Ahlfors has studied the curvature properties of the Weil-Petersson metric. In particular, he proved that the Bers coordinates on $T(\Gamma)$ are geodesic at the origin, and computed a Riemann curvature tensor of the Weil-Petersson metric in terms of multiple principal value integrals. Using these formulas, Ahlfors proved that $T(\Gamma)$ has negative Ricci, holomorphic sectional, and scalar curvatures. Further results have been obtained by Royden [Roy75]. Wolpert re-examined Ahlfors' approach in [Wol86]. He developed a different method for computing Riemann and Ricci curvature tensors, and obtained explicit formulas in terms of the resolvent kernel of the Laplace operator of the hyperbolic metric on the corresponding Riemann surface.

Curvature properties of the infinite-dimensional complex Fréchet manifold $\text{Möb}(S^1)\backslash\text{Diff}_+(S^1)$ have been studied by Kirillov and Yuriev [KY87], and by Bowick and Rajeev [BR87a, BR87b]. In particular, they computed the Riemann curvature tensor of the right-invariant Kähler metric and proved that $\text{Möb}(S^1)\backslash\text{Diff}_+(S^1)$ is a Kähler -Einstein manifold.

Since both the finite-dimensional Teichmüller spaces $T(\Gamma)$ and the homogeneous space $\text{Möb}(S^1)\backslash \text{Diff}_+(S^1)$ are complex submanifolds of $T(1)$, it is natural to investigate whether the latter space carries a "universal" Kähler metric which can be pulled back to the submanifolds. The immediate difficulty is that the universal Teichmüller space $T(1)$ is a complex Banach manifold, so that its tangent spaces do not carry Hermitian metric. Nag and Verjovsky [NV90] were the first to address this problem. They have shown that the Kähler metric on $\text{Möb}(S^1)\backslash \text{Diff}_+(S^1)$ is a pull-back of a certain Hermitian metric defined on a Hilbert subspace of the tangent space at the origin of $T(1)$. The latter metric is analogous to the Weil-Petersson metric on finite-dimensional Teichmüller spaces. However, finite-dimensional Teichmüller spaces $T(\Gamma)$ embed into $T(1)$ transversally to the Hilbert subspace, so that the Weil-Petersson metric on $T(\Gamma)$ can not be pulled back from $T(1)$. Nevertheless, following a suggestion by Velling, Nag and Verjovsky [NV90] have shown that the Weil-Petersson metric on $T(\Gamma)$ can be obtained by a certain "averaging" procedure using Patterson's uniform distribution of the "lattice points" of a cofinite Fuchsian group Γ in the hyperbolic plane. The major open problem is to define the Weil-Petersson metric on the whole space $T(1)$, to study its curvature properties, and to find relation between curvatures of this metric and the Weil-Petersson metric on finite-dimensional Teichmüller spaces [1].

An attempt to define the Weil-Petersson metric on the universal Teichmüller space based on the completion of $\text{diff}(S^1)/\text{möb}(S^1)$ [2] in the Sobolev's 3/2-norm was made in [STZ99]. However, the paper [STZ99] does not contain a rigorous proof that is needed for introducing a Hilbert manifold structure on an infinite-dimensional manifold. Also, the identification between the tangent space $\text{diff}(S^1)/\text{möb}(S^1)$ and the space of holomorphic functions on the unit disk made in [STZ99] is not correct and actually introduces Sobolev's 9/2-norm rather than 3/2-norm. As the result, the corresponding quasi-symmetric homeomorphisms of S^1 are of class $C^3(S^1)$.

Here we introduce Weil-Petersson metric on the universal Teichmüller space $T(1)$ and study its curvature properties. We prove that $T(1)$ carries a new structure of a Hilbert manifold (in the underlying topology $T(1)$ has uncountably many components), and we define the Weil-Petersson metric on $T(1)$ by Hilbert space inner products on tangent spaces. We re-examine the Ahlfors original computation [Ahl62] of the second variation of the hyperbolic metric and of the Riemann tensor for the finite-dimensional Teichmüller spaces in terms of the principal value integrals. We show how to extend the Ahlfors' method to the case of the universal Teichmüller space and how to convert formulas using principal value integrals into closed expressions using resolvent kernel of the Laplace operator on the hyperbolic

[1]See the remark on p. 136 in [NV90].

[2]Here $\text{diff}(S^1)$ and $\text{möb}(S^1)$ are Lie algebras of Lie groups $\text{Diff}_+(S^1)$ and $\text{Möb}(S^1)$.

plane. Our results extend the Wolpert's formulas [Wol86] to the infinite-dimensional Hilbert manifold $T(1)$. We also prove that $T(1)$ is a Kähler-Einstein manifold with negative Ricci and sectional curvatures. Using the averaging procedure, we derive Wolpert's curvature formulas [Wol86] for the finite-dimensional Teichmüller spaces from the curvature formulas for the universal Teichmüller space. Finally, we introduce and compute Mumford-Morita-Miller characteristic forms for the vertical tangent bundle associated with the fibration $\pi : \mathcal{T}(1) \to T(1)$, where $\mathcal{T}(1)$ is the universal Teichmüller curve. Here again we consider $T(1)$ and $\mathcal{T}(1)$ as Hilbert manifolds and show that the integration over the fibers operation, used in the definition of Mumford-Morita-Miller characteristic forms, is well-defined.

This is the first paper in a series. In the subsequent paper we will construct a Kähler potential for the Weil-Petersson metric on $T(1)$ and will study the properties of the period mapping.

Here is the more detailed content of the paper. In Section 2 we present necessary facts from Teichmüller theory, mainly following classical monographs by Ahlfors [Ahl87], Lehto [Leh87] and Nag [Nag88]. Namely, in Section 2.1 we briefly cover: the main definitions, the group structure of the universal Teichmüller space $T(1)$, the Bers embedding, structure of $T(1)$ as an infinite-dimensional complex Banach manifold modeled on the complex Banach space $A_\infty(\mathbb{D})$, and the basic properties of the universal Teichmüller curve $\pi : \mathcal{T}(1) \to T(1)$. In Section 2.2 we realize $T(1)$ and $\mathcal{T}(1)$ as homogeneous spaces of the group $\mathrm{Homeo}_{qs}(S^1)$ of quasi-symmetric homeomorphisms of S^1, and by using conformal welding we identify $T(1)$ and $\mathcal{T}(1)$ with the spaces of univalent functions on the unit disk $\mathbb{D}$. We describe the decomposition of the tangent bundle of $\mathcal{T}(1)$ over the fiber $\pi^{-1}(0)$ and present isomorphisms between the tangent spaces. Lemma 2.5 describing a special property of the quasiconformal mapping with harmonic Beltrami differential seems to be a new result. In Section 2.3 we present, in a succinct form, basic facts about the Teichmüller spaces and Teichmüller curves of Fuchsian groups, including the definition of the Weil-Petersson metric, and Patterson's lemma on the uniform distribution of lattice points on the hyperbolic plane. In Section 2.4 we collect necessary properties of the resolvent kernel $G = \frac{1}{2}(\Delta_0 + \frac{1}{2})^{-1}$ of the Laplace operator Δ_0 on the hyperbolic plane, and in Section 2.5 we present Ahlfors' classical variational formulas. In Section 3 we introduce new Hilbert manifold structure on $T(1)$. Namely, in Section 3.1 we define the Hilbert subspaces $H^{-1,1}(\mathbb{D}^*)$ and $A_2(\mathbb{D})$ of the tangent spaces to $T(1)$ and to $A_\infty(\mathbb{D})$. In Theorem we 3.3 prove that the differential of the Bers embedding $\beta : T(1) \to A_\infty(\mathbb{D})$ is a bounded bijection between these Hilbert spaces. In Section 3.2 we prepare all L^2-estimates used in Section 3.3. The main result there is Theorem 3.10 — the existence of the Hilbert manifold atlas for $T(1)$. In Theorem 3.13 we prove that the Bers embedding is also a biholomorphic mapping of Hilbert manifolds. In Section 4.1, following [Teo02], we recall the definition of the Velling-Kirillov metric on the universal Teichmüller curve $\mathcal{T}(1)$ considered as a Banach manifold, and in Section 4.2 we define the Weil-Petersson metric on the Hilbert manifold $T(1)$. In Section 5.1 we prove that Velling-Kirillov metric is real-analytic on $\mathcal{T}(1)$ by explicitly constructing its real-analytic Kähler potential — Theorem 5.3. We introduce Mumford-Miller-Morita characteristic forms by considering $\pi : \mathcal{T}(1) \to T(1)$ as fibration of Hilbert manifolds. The latter property is crucial for the operation "integration over the fibers" (which are non-compact) to be well-defined. In Theorem 5.10 we explicitly compute Mumford-Miller-Morita forms in terms of the resolvent G. This is an infinite-dimensional generalization of Wolpert's result in [Wol86]. In Section 6 we give a simple derivation of the second variation of the hyperbolic metric — Proposition 6.3. In Section 7 we prove that the Weil-Petersson metric on $T(1)$ is Kähler and explicitly compute its Riemann and Ricci curvature tensors, showing that $T(1)$ is a Kähler -Einstein manifold. The main results there are Theorem 7.7 and 7.11. They are based on a more technical Proposition 7.2 and Lemma 7.8, and the proof of the latter is presented in the Appendix. Finally, in Section 8 we derive Wolpert's curvature formulas [Wol86] for finite-dimensional Teichmüller spaces from the corresponding "universal" curvature formulas for $T(1)$, obtained in Section 7.

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2. The universal Teichmüller space

2.1. **Teichmüller theory.** Here we present, in a succinct form, necessary facts from the Teichmüller theory (for more details, see monographs [Ahl87, Leh87, Nag88] and the exposition in [Teo02]).

2.1.1. *Main definitions.* Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and let $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}$ be its exterior. Denote by $L^\infty(\mathbb{D}^*)$ and $L^\infty(\mathbb{D})$ the complex Banach spaces of bounded Beltrami differentials on $\mathbb{D}^*$ and $\mathbb{D}$ respectively, and let $L^\infty(\mathbb{D}^*)_1$ be the open unit ball in $L^\infty(\mathbb{D}^*)$. Two classical models of the universal Teichmüller space $T(1)$ are the following.

Model A. Extend every $\mu \in L^\infty(\mathbb{D}^*)_1$ to $\mathbb{D}$ by the reflection

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)}\frac{z^2}{\bar{z}^2}\ ,\ z \in \mathbb{D}, \tag{2.1}$$

and consider the unique quasiconformal (q.c.) mapping $w_\mu : \mathbb{C} \to \mathbb{C}$, which fixes $-1, -i$ and 1 (i.e., is normalized) and satisfies the Beltrami equation

$$(w_\mu)_{\bar{z}} = \mu(w_\mu)_z\ .$$

Here and in what follows subscripts z and $\bar{z}$ always stand for the partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, unless it is explicitly stated otherwise. Due to the reflection symmetry (2.1) the q.c. mapping w_μ satisfies

$$\frac{1}{w_\mu(z)} = \overline{w_\mu\left(\frac{1}{\bar{z}}\right)} \tag{2.2}$$

and fixes domains $\mathbb{D}$, $\mathbb{D}^*$, and the unit circle S^1. For $\mu, \nu \in L^\infty(\mathbb{D}^*)_1$ set $\mu \sim \nu$ if $w_\mu|_{S^1} = w_\nu|_{S^1}$. The universal Teichmüller space $T(1)$ is defined as a set of equivalence classes of normalized q.c. mappings w_μ,

$$T(1) = L^\infty(\mathbb{D}^*)_1/\sim\ .$$

Model B. Extend every $\mu \in L^\infty(\mathbb{D}^*)_1$ to be zero outside $\mathbb{D}^*$, and consider the unique q.c. mapping w^μ which satisfies the Beltrami equation

$$w^\mu_{\bar{z}} = \mu w^\mu_z,$$

and is normalized by the conditions $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$. Here $f = w^\mu|_{\mathbb{D}}$ is holomorphic on $\mathbb{D}$ and prime stands for the derivative. For $\mu, \nu \in L^\infty(\mathbb{D}^*)_1$ set $\mu \sim \nu$ if $w^\mu|_{\mathbb{D}} = w^\nu|_{\mathbb{D}}$. The universal Teichmüller space $T(1)$ is defined as a set of equivalence classes of normalized q.c. mappings w^μ,

$$T(1) = L^\infty(\mathbb{D}^*)_1/\sim\ .$$

Since $w_\mu|_{S^1} = w_\nu|_{S^1}$ if and only if $w^\mu|_{\mathbb{D}} = w^\nu|_{\mathbb{D}}$, these two definitions of the universal Teichmüller space are equivalent. The set $T(1)$ is a topological space with the quotient topology induced from $L^\infty(\mathbb{D}^*)_1$. Denote by $\mathcal{L}^\infty(\mathbb{D}^*)$ the subspace of $L^\infty(\mathbb{D}^*)$ consisting of real-analytic Beltrami differentials. Every point in $T(1)$ can be represented by $\mu \in \mathcal{L}^\infty(\mathbb{D}^*)$ [Leh87, Sect. III.1.1].

The space $T(1)$ has a unique structure of a complex Banach manifold, such that the projection map

$$\Phi : L^\infty(\mathbb{D}^*)_1 \to T(1)$$

is a holomorphic submersion. The differential of Φ at the origin

$$D_0\Phi : L^\infty(\mathbb{D}^*) \to T_0T(1)$$

is a complex linear surjection of holomorphic tangent spaces. The kernel of $D_0\Phi$ is the subspace $\mathcal{N}(\mathbb{D}^*)$ of infinitesimally trivial Beltrami differentials. Explicitly,

$$\mathcal{N}(\mathbb{D}^*) = \left\{ \mu \in L^\infty(\mathbb{D}^*) : \iint\limits_{\mathbb{D}^*} \mu\, \phi\, d^2z = 0 \text{ for all } \phi \in A_1(\mathbb{D}^*) \right\},$$

where $d^2z = dx \wedge dy$, $z = x + iy$, and

$$A_1(\mathbb{D}^*) = \left\{ \phi \text{ holomorphic on } \mathbb{D}^* : \iint\limits_{\mathbb{D}^*} |\phi| d^2z < \infty \right\}.$$

The Banach space of bounded harmonic Beltrami differentials on $\mathbb{D}^*$ is defined by

$$\Omega^{-1,1}(\mathbb{D}^*) = \left\{\mu \in L^\infty(\mathbb{D}^*) : \mu(z) = (1-|z|^2)^2\overline{\phi(z)},\ \phi \in A_\infty(\mathbb{D}^*)\right\},$$

where

$$A_\infty(\mathbb{D}^*) = \left\{\phi \text{ holomorphic on } \mathbb{D}^* : \|\phi\|_\infty = \sup_{z\in\mathbb{D}^*}\left|(1-|z|^2)^2\phi(z)\right| < \infty\right\}.$$

The Banach space $\Omega^{-1,1}(\mathbb{D}^*)$ is not separable. The decomposition

$$L^\infty(\mathbb{D}^*) = \mathcal{N}(\mathbb{D}^*) \oplus \Omega^{-1,1}(\mathbb{D}^*) \tag{2.3}$$

identifies the holomorphic tangent space $T_0T(1) = L^\infty(\mathbb{D}^*)/\mathcal{N}(\mathbb{D}^*)$ at the origin of $T(1)$ with the Banach space $\Omega^{-1,1}(\mathbb{D}^*)$. The universal Teichmüller space $T(1)$ is a complex Banach manifold modeled on $\Omega^{-1,1}(\mathbb{D}^*)$.

Remark 2.1. Traditionally, the universal Teichmüller space is defined using the complex Banach space $L^\infty(\mathbb{D})_1$. The reflection (2.1) establishes natural complex anti-linear isomorphism between $L^\infty(\mathbb{D}^*)_1$ and $L^\infty(\mathbb{D})_1$, and the universal Teichmüller space in the traditional definition is complex conjugate to the space $T(1)$ defined above.

2.1.2. *The group structure.* The unit ball $L^\infty(\mathbb{D}^*)_1$ carries a group structure induced by the composition of q.c. mappings. The group law

$$\lambda = \nu * \mu^{-1}$$

is defined through $w_\lambda = w_\nu \circ w_\mu^{-1}$, where μ^{-1} stands for the inverse element to μ, i.e., $\mu * \mu^{-1} = 0$. The group law is given explicitly by

$$\lambda = \left(\frac{\nu-\mu}{1-\bar{\mu}\nu}\,\frac{(w_\mu)_z}{(\overline{w}_\mu)_{\bar{z}}}\right)\circ w_\mu^{-1}.$$

It follows from this formula that $\mathcal{L}^\infty(\mathbb{D}^*)_1$ is a subgroup of $L^\infty(\mathbb{D}^*)_1$.

For every $\lambda \in L^\infty(\mathbb{D}^*)_1$ set $[\lambda] = \Phi(\lambda) \in T(1)$. The group structure on $L^\infty(\mathbb{D}^*)_1$ projects to $T(1)$ by $[\lambda]*[\mu] = [\lambda*\mu]$. For every $\mu \in L^\infty(\mathbb{D}^*)_1$ the right translations

$$R_{[\mu]} : T(1) \to T(1), \quad [\lambda] \mapsto [\lambda*\mu],$$

are biholomorphic automorphisms of $T(1)$. The left translations, in general, are not even continuous mappings (see, e.g., [Leh87, Sect. III.3.4]). For every $\mu \in L^\infty(\mathbb{D}^*)_1$ the kernel of $D_\mu\Phi$ is the subspace $D_0R_\mu\left(\mathcal{N}(\mathbb{D}^*)\right)$ of $L^\infty(\mathbb{D}^*)$ and

$$T_{[\mu]}T(1) = D_0R_{[\mu]}\left(T_0T(1)\right) \simeq D_0R_\mu\left(\Omega^{-1,1}(\mathbb{D}^*)\right).$$

2.1.3. *The Bers embedding.* Let $A_\infty(\mathbb{D})$ be the complex Banach space

$$A_\infty(\mathbb{D}) = \left\{ \phi \text{ holomorphic on } \mathbb{D} : \|\phi\|_\infty = \sup_{z\in\mathbb{D}} \left|(1-|z|^2)^2\phi(z)\right| < \infty \right\}.$$

The Bers embedding $\beta : T(1) \hookrightarrow A_\infty(\mathbb{D})$ is defined as follows. Denote by $\mathcal{S}(f)$ the Schwarzian derivative of a conformal map f,

$$\mathcal{S}(f) = \frac{f_{zzz}}{f_z} - \frac{3}{2}\left(\frac{f_{zz}}{f_z}\right)^2.$$

For every $\mu \in L^\infty(\mathbb{D}^*)_1$ the holomorphic function $\mathcal{S}(w^\mu)|_{\mathbb{D}} \in A_\infty(\mathbb{D})$ (by Kraus-Nehari inequality it lies in the ball of radius 6 in $A_\infty(\mathbb{D})$). Set

$$\beta([\mu]) = \mathcal{S}(w^\mu|_{\mathbb{D}}).$$

The Bers embedding is a holomorphic map of complex Banach manifolds, and its differential at the origin is

$$D_0\beta(\mu)(z) = -\frac{6}{\pi}\iint\limits_{\mathbb{D}^*} \frac{\mu(\zeta)}{(\zeta - z)^4} d^2\zeta. \tag{2.4}$$

The complex-linear mapping $D_0\beta$ induces the isomorphism $\Omega^{-1,1}(\mathbb{D}^*) \xrightarrow{\sim} A_\infty(\mathbb{D})$ of the holomorphic tangent spaces to $T(1)$ and $A_\infty(\mathbb{D})$ at the origin. The mapping $\Lambda : A_\infty(\mathbb{D}) \to \Omega^{-1,1}(\mathbb{D}^*)$, inverse to $D_0\beta$, is given by

$$\mu(z) = \Lambda(\phi)(z) = -\tfrac{1}{2}(1-|z|^2)^2\phi\left(\tfrac{1}{\bar{z}}\right)\tfrac{1}{\bar{z}^4}.$$

According to the Ahlfors-Weill theorem, over the ball of radius 2 in $A_\infty(\mathbb{D})$ the map $\phi \mapsto [\Lambda(\phi)]$ is the right inverse to β, $\beta \circ \Lambda = \mathrm{id}$.

2.1.4. *The complex structure.* For every $\mu \in L^\infty(\mathbb{D}^*)_1$ let $U_\mu \subset T(1)$ be the image of the ball of radius 2 in $A_\infty(\mathbb{D})$ under the map $h_\mu^{-1} = \Phi \circ R_\mu \circ \Lambda$. The maps $h_{\mu\nu} = h_\mu \circ h_\nu^{-1} : h_\nu(U_\mu \cap U_\nu) \to h_\mu(U_\mu \cap U_\nu)$ are biholomorphic as functions on the Banach space $A_\infty(\mathbb{D})$. The structure of $T(1)$ as a complex Banach manifold modeled on the Banach space $A_\infty(\mathbb{D})$ is explicitly described by the complex-analytic atlas given by the open covering

$$T(1) = \bigcup_{\mu\in L^\infty(\mathbb{D}^*)_1} U_\mu$$

with coordinate maps h_μ and transition maps $h_{\mu\nu}$. The canonical projection $\Phi : L^\infty(\mathbb{D}^*)_1 \to T(1)$ is a holomorphic submersion and the Bers embedding $\beta : T(1) \to A_\infty(\mathbb{D})$ is a biholomorphic map with respect to this complex structure.

Remark 2.2. Since every point $T(1)$ can be represented by a real-analytic Beltrami differential, it is sufficient to consider the atlas formed by the charts (U_μ, h_μ) with $\mu \in \mathcal{L}^\infty(\mathbb{D}^*)_1$.

Complex coordinates on $T(1)$ defined by the coordinate charts (U_μ, h_μ) are called Bers coordinates. For every $\nu \in \Omega^{-1,1}(\mathbb{D}^*)$ set $\phi = D_0\beta(\nu)$ and define a holomorphic vector field $\frac{\partial}{\partial\varepsilon_\nu}$ on U_0 by setting

$$Dh_0\left(\tfrac{\partial}{\partial\varepsilon_\nu}\right) = \phi$$

at all points in U_0 [3]. At every point $[\mu] \in U_0$, identified with the corresponding harmonic Beltrami differential μ, the vector field $\frac{\partial}{\partial\varepsilon_\nu}$ in terms of the Bers coordinates on U_μ corresponds to

$$\tilde{\phi} = D_\mu h_\mu\left(\tfrac{\partial}{\partial\varepsilon_\nu}\right) = \left(D_\mu h_\mu\,(D_\mu h_0)^{-1}\right)(\phi) = D_0\,(\beta\circ\Phi)\left(D_\mu R_\mu^{-1}\,(\Lambda(\phi))\right).$$

Using identification $\Omega^{-1,1}(\mathbb{D}^*) \simeq A_\infty(\mathbb{D})$, provided by mapping $D_0\beta$, we get

$$\left.\frac{\partial}{\partial\varepsilon_\nu}\right|_\mu = P\left(D_\mu R_\mu^{-1}(\nu)\right) = P\left(R(\nu,\mu)\right), \tag{2.5}$$

where

$$R(\nu,\mu) = \left(\frac{\nu}{1-|\mu|^2}\,\frac{(w_\mu)_z}{(\overline{w}_\mu)_{\bar z}}\right)\circ w_\mu^{-1}, \tag{2.6}$$

and $P : L^\infty(\mathbb{D}^*) \to \Omega^{-1,1}(\mathbb{D}^*)$ is projection onto the subspace of harmonic Beltrami differentials, defined by the decomposition (2.3). Explicitly,

$$(P\mu)(z) = \frac{3(1-|z|^2)^2}{\pi}\iint\limits_{\mathbb{D}^*}\frac{\mu(\zeta)}{(1-\zeta\bar z)^4}d^2\zeta. \tag{2.7}$$

Remark 2.3. Right translating $\nu \in T_0T(1)$ defines a holomorphic tangent vector

$$D_0R_{[\mu]}(\nu) = (1-|\mu|^2)\,\nu\circ w_\mu\,\frac{(\overline{w}_\mu)_{\bar z}}{(w_\mu)_z} \in T_{[\mu]}T(1)$$

at every $[\mu] \in T(1)$. In Bers coordinates on U_μ this tangent vector is represented by $\nu \in \Omega^{-1,1}(\mathbb{D}^*)$. However, the family $\{D_0R_{[\mu]}(\nu)\}_{[\mu]\in T(1)}$ of holomorphic tangent vectors does not form a smooth vector field on $T(1)$ since the left translations are not continuous on $T(1)$.

2.1.5. *The universal Teichmüller curve.* The universal Teichmüller curve $\mathcal{T}(1)$ is a natural complex fiber space over $T(1)$ with a holomorphic projection map $\pi : \mathcal{T}(1) \to T(1)$. The fiber over each point $[\mu]$ is a quasi-disc $w^\mu(\mathbb{D}^*) \subset \hat{\mathbb{C}}$ with complex structure induced from $\hat{\mathbb{C}}$ and

$$\mathcal{T}(1) = \{([\mu], z) : [\mu] \in T(1),\ z \in w^\mu(\mathbb{D}^*)\}. \tag{2.8}$$

The fibration $\pi : \mathcal{T}(1) \to T(1)$ has a natural holomorphic section given by $T(1) \ni [\mu] \mapsto ([\mu], \infty) \in \mathcal{T}(1)$ — "zero section", which defines the embedding $T(1) \hookrightarrow \mathcal{T}(1)$. The universal Teichmüller curve is a complex Banach

[3] We identify holomorphic tangent space to $A_\infty(\mathbb{D})$ at every point with $A_\infty(\mathbb{D})$.

manifold modeled on $A_\infty(\mathbb{D}) \oplus \mathbb{C}$ [4], and the mapping

$$T(1) \times \mathbb{D}^* \ni ([\mu], z) \mapsto ([\mu], w^\mu(z)) \in \mathcal{T}(1)$$

is a real-analytic isomorphism.

2.2. **Homogeneous spaces of $\mathbf{Homeo}_{qs}(S^1)$.** Let $\mathrm{Homeo}_{qs}(S^1)$ be the group of orientation preserving quasi-symmetric homeomorphisms of the unit circle S^1 (see, e.g., [Leh87] for the definition), and let $\mathrm{Diff}_+(S^1)$, $\mathrm{M\ddot{o}b}(S^1)$, and S^1 be the subgroups of $\mathrm{Homeo}_{qs}(S^1)$ consisting, respectively, of smooth orientation preserving diffeomorphisms of S^1, of Möbius transformations of S^1, and of rotations of S^1.

Denote by $\mathcal{U}$ the set of univalent functions on $\mathbb{D}$ and let

$$\mathcal{D} = \left\{ f \in \mathcal{U} : f(0) = 0, f'(0) = 1, f''(0) = 0, f \text{ admits a q.c. extension to } \mathbb{C} \right\},$$

$$\widetilde{\mathcal{D}} = \left\{ f \in \mathcal{U} : f(0) = 0, f'(0) = 1, f \text{ admits a q.c. extension to } \mathbb{C} \right\}.$$

According to the Beurling-Ahlfors extension theorem, the maps

$$T(1) \ni [\mu] \mapsto w^\mu|_{\mathbb{D}} \in \mathcal{D}$$

and

$$T(1) \ni [\mu] \mapsto w_\mu|_{S^1} \in \mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)$$

define bijections

$$\mathcal{D} \xleftarrow{\sim} T(1) \xrightarrow{\sim} \mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1), \tag{2.9}$$

which endow the spaces $\mathcal{D}$ and $\mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)$ with the structure of complex Banach manifolds modeled on the Banach space $A_\infty(\mathbb{D})$. In what follows, we will always identify the coset space $\mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)$ with the subgroup of $\mathrm{Homeo}_{qs}(S^1)$ fixing $1, -1$ and i, so that the bijection $T(1) \xrightarrow{\sim} \mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)$ is a group isomorphism.

Remark 2.4. It is a non-trivial problem to describe the complex Banach manifold structure of the spaces $\mathcal{D}$ and $\mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)$ intrinsically, without using the bijection (2.9).

2.2.1. *Conformal welding.* According to the Beurling-Ahlfors extension theorem, for every $\gamma \in \mathrm{M\ddot{o}b}\, S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ there exists unique $\alpha \in \mathrm{M\ddot{o}b}(S^1)$ which fixes 1, and univalent functions f and g on $\mathbb{D}$ and $\mathbb{D}^*$, satisfying the following properties.

CW1. f and g admit q.c. extensions to $\mathbb{C}$.
CW2. $\alpha \circ \gamma = (g^{-1} \circ f)|_{S^1}$.
CW3. $f(0) = 0,\ f'(0) = 1,\ f''(0) = 0$.
CW4. $g(\infty) = \infty$.

[4]Here $\hat{\mathbb{C}} \setminus \{0\}$ is identified with $\mathbb{C}$ via the conformal map $z \mapsto 1/z$.

The factorization **CW2** is known as conformal welding. For $\gamma = w_\mu|_{S^1}$, $[\mu] \in T(1)$, $f = w^\mu|_{\mathbb{D}}$ and $g = (w^\mu \circ w_\mu^{-1} \circ \alpha^{-1})|_{\mathbb{D}^*}$, so that $g(\mathbb{D}^*) = w^\mu(\mathbb{D}^*)$. Here w^μ is normalized so that f satisfies **CW3** and $\alpha \in \text{Möb}(S^1)$ is uniquely determined by the conditions $\alpha(1) = 1$ and **CW4**. For $[\mu] \in T(1)$ we will always denote $\gamma_\mu = (\alpha \circ w_\mu)|_{S^1}$, $f^\mu = f$ and $g_\mu = g$, so that

$$\gamma_\mu = (g_\mu^{-1} \circ f^\mu)|_{S^1}.$$

Slightly abusing notations, we will denote by γ_μ a q.c. extension of $\gamma_\mu = (\alpha \circ w_\mu)|_{S^1} \in \text{Möb}(S^1)\backslash\text{Homeo}_{qs}(S^1)$ given by $\alpha \circ w_\mu$. Since $\alpha \in \text{Möb}(S^1)$ fixes 1, the q.c. mapping γ_μ satisfies reflection property (2.2) and the factorization

$$\gamma_\mu = g_\mu^{-1} \circ f^\mu, \tag{2.10}$$

where $f^\mu = w^\mu$ and $g_\mu = w^\mu \circ w_\mu^{-1} \circ \alpha^{-1}$. We will distinguish between $\gamma_\mu \in \text{Möb}(S^1)\backslash\text{Homeo}_{qs}(S^1)$ and its q.c. extension by explicitly specifying either the property **CW2** or the factorization (2.10).

The following result will be used in Section 3.

Lemma 2.5. *Let $\gamma_\mu = \alpha \circ w_\mu$ be the q.c. mapping introduced above. Then for every $\mu \in \Omega^{-1,1}(\mathbb{D}^*)_1 = \Omega^{-1,1}(\mathbb{D}^*) \cap L^\infty(\mathbb{D}^*)_1$ the mapping γ_μ fixes 0 and ∞.*

Proof. By the reflection property (2.2) and the factorization (2.10), it is sufficient to prove that $f^\mu = w^\mu$ fixes ∞. Denote

$$\gamma = j \circ \gamma_\mu \circ j, \quad g = j \circ g_\mu \circ j, \quad f = j \circ f^\mu \circ j, \quad j^*(\mu) = \mu \circ j \, \frac{\overline{j_z}}{j_z},$$

where $j(z) = z^{-1}$. The factorization (2.10) for γ_μ gives $\gamma = g^{-1} \circ f$, and the property **CW3** for f^μ yields the following Laurent expansion of f at ∞,

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots . \tag{2.11}$$

We will prove that $f(0) = 0$ for $\mu \in \Omega^{-1,1}(\mathbb{D}^*)_1$ by exploiting the argument in Royden-Earle's proof of the Ahlfors-Weill theorem, as presented in [Nag88, Sect. 3.8.5].

Namely, f satisfies the Beltrami equation with the Beltrami differential $\nu = j^*(\mu)|_{\mathbb{D}}$, which is supported on $\mathbb{D}$. The fundamental theorem from the theory of q.c. mappings (see, e.g. [Ahl87]) asserts that f admits the series representation

$$f(z) = z + P(\nu)(z) + P(\nu H(\nu))(z) + P(\nu H(\nu H(\nu)))(z) + \cdots , \tag{2.12}$$

which is uniformly and absolutely convergent on $\mathbb{C}$. Here for $h \in C^2(\mathbb{D})$ we denote

$$P(h)(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{h(\zeta)}{\zeta - z}\, d^2\zeta,$$

$$H(h)(\zeta) = -\frac{1}{\pi} \iint\limits_{\mathbb{D}} \frac{h(\zeta)}{(\zeta - z)^2}\, d^2\zeta,$$

where the latter integral — the Hilbert transform, is understood in the principal value sense. Since ν has compact support, it immediately follows from the definition of the operators P and H that the series (2.12) has the Laurent expansion (2.11) at ∞. We will prove that for $\nu \in \Omega^{-1,1}(\mathbb{D})$ each term of this series vanishes at $z = 0$. Representing $\nu(z) = -\frac{1}{2}(1 - |z|^2)^2 \sum_{n=0}^{\infty} a_n \bar{z}^n$ and using polar coordinates, we get for any $(n-1)$ – iterate of the operator νH, $n > 1$,

$$\begin{aligned} &P(\nu H(\nu H(\nu \dots H(\nu))))(0) \\ &= \left(\frac{1}{2\pi}\right)^n \iint\limits_{\mathbb{D}} \cdots \iint\limits_{\mathbb{D}} \frac{\sum_{m_1,\dots m_n} a_{m_1} \dots a_{m_n} r_1^{m_1+1} \dots r_n^{m_n+1} e^{-im_1\theta_1} \dots e^{-im_n\theta_n}}{r_1 e^{i\theta_1}(r_2 e^{i\theta_2} - r_1 e^{i\theta_1})^2 \dots (r_n e^{i\theta_n} - r_{n-1} e^{i\theta_{n-1}})^2} \\ &\qquad (1 - r_1^2)^2 \dots (1 - r_n^2)^2 dr_1 d\theta_1 \dots dr_n d\theta_n \\ &= \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1} \dots a_{m_n} I_{m_1,\dots,m_n}, \end{aligned}$$

where each integral in the definition of H is understood in the principal value sense. The interchange of the orders of the summation and integration can be easily justified. For fixed $r_1 \neq 0, r_1 \neq r_2, r_2 \neq r_3, \dots, r_{n-1} \neq r_n$, let

$$\begin{aligned} &I_{m_1,\dots,m_n}(r_1, \dots, r_n) \\ &= \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{e^{-im_1\theta_1} \dots e^{-im_n\theta_n} d\theta_1 \cdots d\theta_n}{r_1 e^{i\theta_1}(r_2 e^{i\theta_2} - r_1 e^{i\theta_1})^2 \dots (r_n e^{i\theta_n} - r_{n-1} e^{i\theta_{n-1}})^2}. \end{aligned}$$

A change of variables $\theta_k \mapsto \theta_k + \theta$, $k = 1, \dots, n$ gives

$$I_{m_1,\dots,m_n}(r_1, \dots, r_n) = e^{-i(m_1 + \dots + m_n + (2n-1))\theta} I_{m_1,\dots,m_n}(r_1, \dots, r_n).$$

Since all $m_k \geq 0$ and $2n-1 > 0$ for $n \geq 1$, we have $e^{-i(m_1+\dots+m_n+(2n-1))\theta} \neq 1$ and hence

$$I_{m_1,\dots,m_n}(r_1, \dots, r_n) = 0.$$

This proves that all $I_{m_1,\dots,m_n}$ vanish and, therefore, $f(0) = 0$. □

Remark 2.6. Since $P(f)_z = H(f)$, it also follows from the proof that $f_z(0) = 1$.

Similar to (2.9) , there are bijections

$$\widetilde{\mathcal{D}} \xleftarrow{\sim} \mathcal{T}(1) \xrightarrow{\sim} S^1 \backslash \mathrm{Homeo}_{qs}(S^1),$$

where we always identify the coset space $S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ with the stabilizer of 1 in $\mathrm{Homeo}_{qs}(S^1)$ (see, e.g., [Teo02]). For every $\gamma \in S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ there exist unique univalent functions f and g on $\mathbb{D}$ and $\mathbb{D}^*$, satisfying the properties **CW1, CW4** and

CW2′. $\gamma = (g^{-1}\circ f)|_{S^1}$;
CW3′. $f(0)=0$, $f'(0)=1$.

Namely, the fibration $\pi : \mathcal{T}(1) \longrightarrow T(1)$ corresponds to the fiber space $S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ over $\mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)$ with the fibers isomorphic to $S^1\backslash\mathrm{M\ddot{o}b}(S^1)\simeq\mathbb{D}^*$. The points in the fiber at $[\mu]\in T(1)$ correspond to the points $\sigma_w\circ\gamma_\mu \in S^1\backslash\mathrm{Homeo}_{qs}(S^1)$, $w\in\mathbb{D}^*$ with[5]

$$\sigma_w(z) = \frac{1-w}{1-\bar{w}}\,\frac{1-z\bar{w}}{z-w} \in S^1\backslash\mathrm{M\ddot{o}b}(S^1).$$

Using property **CW2** for γ_μ, we get the factorization **CW2′**,

$$\gamma = \sigma_w\circ\gamma_\mu = (g^{-1}\circ f)|_{S^1},$$

where

$$f = \lambda_w\circ f^\mu, \quad g = \lambda_w\circ g_\mu\circ\sigma_w^{-1},$$

and

$$\lambda_w(z) = \frac{z}{c_w z+1}, \quad c_w = -\frac{1}{g_\mu(w)}.$$

Since $(g_\mu\circ\sigma_w^{-1})(\infty) = g_\mu(w)$, the functions f and g satisfy the properties **CW3′** and **CW4** respectively, and the mapping

$$\mathcal{T}(1) \ni ([\mu], g_\mu(w)) \mapsto \gamma = \sigma_w\circ\gamma_\mu \in S^1\backslash\mathrm{Homeo}_{qs}(S^1)$$

establishes the isomorphism $\mathcal{T}(1) \xrightarrow{\sim} S^1\backslash\mathrm{Homeo}_{qs}(S^1)$.

As before, we will also denote by γ a q.c. extension of $\gamma\in S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ which satisfies the reflection property (2.2) and admits the factorization $\gamma = g^{-1}\circ f$.

Remark 2.7. It is known [Kir87] that $\mathrm{Diff}_+(S^1)$ is an infinite-dimensional Lie group and homogeneous spaces $\mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Diff}_+(S^1)$ and $S^1\backslash\mathrm{Diff}_+(S^1)$ are infinite-dimensional complex Fréchet manifolds. In this case conformal welding readily follows from the Riemann mapping theorem without using q.c. mappings [Kir87]. Note that our convention for the conformal welding is different from that in [Kir87]: we are using right cosets instead of left cosets.

The bijection $\mathcal{T}(1) \xrightarrow{\sim} S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ endows the universal Teichmüller curve $\mathcal{T}(1)$ with the group structure. Explicitly,

$$([\lambda], z) = ([\nu], \zeta) * ([\mu], w)^{-1},$$

[5]Here the subscript w does not stand for the derivative.

where

$$\lambda = \left(\frac{\nu - \mu}{1 - \bar{\mu}\nu} \, \frac{\gamma_z}{\overline{\gamma_z}} \right) \circ \gamma^{-1} \tag{2.13}$$

and

$$z = \left(w^{\lambda} \circ \gamma \circ (w^{\nu})^{-1} \right) (\zeta). \tag{2.14}$$

Here γ is a q.c. extension of $\sigma_u \circ \gamma_\mu$, $u = g_\mu^{-1}(w)$, and the point $([\lambda], z) \in \mathcal{T}(1)$ does not depend on the choice of the extension γ.

2.2.2. *The horizontal and vertical subspaces.* The right translations $R_{([\mu],z)} : \mathcal{T}(1) \to \mathcal{T}(1)$ are biholomorphic automorphisms of $\mathcal{T}(1)$ [Ber73]. The holomorphic tangent space to $\mathcal{T}(1)$ at $([\mu], z)$ is identified with the holomorphic tangent space at $(0, \infty)$ — the origin of $\mathcal{T}(1)$ by

$$T_{([\mu],z)}\mathcal{T}(1) = D_{(0,\infty)} R_{([\mu],z)}(T_{(0,\infty)}\mathcal{T}(1)) \simeq T_{(0,\infty)}\mathcal{T}(1).$$

The holomorphic tangent space at the origin naturally splits into the direct sum of horizontal and vertical subspaces,

$$T_{(0,\infty)}\mathcal{T}(1) = \Omega^{-1,1}(\mathbb{D}^*) \oplus \mathbb{C}.$$

The identification of holomorphic tangent spaces provides a natural splitting of the tangent space at every point in $\mathcal{T}(1)$ into the direct sum of horizontal and vertical subspaces. Lifts of horizontal and vertical tangent vectors at the origin of $\mathcal{T}(1)$ to every point in the fiber at the origin are explicitly described as follows.

TV1. Let $\mu \in \Omega^{-1,1}(\mathbb{D}^*) \subset T_{(0,\infty)}\mathcal{T}(1)$ be a horizontal tangent vector to $\mathcal{T}(1)$ at the origin. A curve $([t\mu], z(t))$, $z(0) = z$, which defines the horizontal lift of μ to the point $(0, z) \in \mathcal{T}(1)$ in the fiber $\pi^{-1}(0)$ at the origin, for small t is given by the equation

$$([\mu(t)], \infty) * (0, z) = ([t\mu], z(t)).$$

Using (2.13), (2.14) and Lemma 2.5, we get

$$\mu(t) = (\sigma_z^{-1})^*(t\mu) = t\mu \circ \sigma_z^{-1} \frac{\overline{(\sigma_z^{-1})'}}{(\sigma_z^{-1})'} \quad \text{and} \quad z(t) = w^{t\mu}(z).$$

Thus the horizontal lift of $\mu \in T_{(0,\infty)}\mathcal{T}(1)$ to every point in the fiber $(0, z) \in \pi^{-1}(0)$ is the vector field

$$\tau_\mu = \frac{\partial}{\partial \varepsilon_\mu}\bigg|_0 + \dot{w}^\mu(z) \frac{\partial}{\partial z}, \quad \text{where} \quad \dot{w}^\mu(z) = \frac{dz}{dt}(0)$$

(cf. [Wol86]). At point $(0, z) \in \pi^{-1}(0)$ the vector field τ_μ is identified with the horizontal tangent vector $(\sigma_z^{-1})^*\mu \in T_{(0,\infty)}\mathcal{T}(1)$.

TV2. Let $\mathbf{1} \in \mathbb{C} \subset T_{(0,\infty)}\mathcal{T}(1)$ be the vertical tangent vector to $\mathcal{T}(1)$ at the origin, given by the value of the vector field $\partial_z = \frac{\partial}{\partial z}$ at $z = \infty$.

A curve defining the right translate of $\mathbf{1}$ to the point $(0, z) \in \mathcal{T}(1)$ in the fiber $\pi^{-1}(0)$ at the origin for small t is given by the equation

$$(0, t^{-1}) * (0, z) = (0, z(t)),$$

and it follows from (2.14) that

$$\frac{dz}{dt}(0) = \frac{(1-z)(1-|z|^2)}{(1-\bar{z})}.$$

Thus the right translate of $\mathbf{1} \in T_{(0,\infty)}\mathcal{T}(1)$ to the point $(0, z) \in \pi^{-1}(0)$ is the vector $\frac{(1-z)(1-|z|^2)}{(1-\bar{z})}\partial_z$ at $(0, z)$. As the result, the vector field ∂_z at every point $(0, z) \in \pi^{-1}(0)$ is identified with the vertical tangent vector

$$\frac{(1-\bar{z})}{(1-z)(1-|z|^2)}\mathbf{1} \in T_{(0,\infty)}\mathcal{T}(1).$$

2.2.3. *The isomorphisms of the tangent spaces.* The real tangent vector space $T_0^{\mathbb{R}} S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ to $S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ at the origin is identified with the subspace of Zygmund class continuous real-valued vector fields $\mathrm{u} = u(\theta)\frac{d}{d\theta}$ on S^1 (see, e.g., [Teo02] for the definition), satisfying

$$\int_0^{2\pi} u(\theta)d\theta = 0.$$

In particular, the Fourier series $u(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$ is absolutely convergent. For $|z| = 1$ set

$$\tilde{u}(z) = i\sum_{n\in\mathbb{Z}\backslash\{0\}} c_n z^{n+1}.$$

The function $\tilde{u}$ on S^1 admits the decomposition

$$\tilde{u} = u_+ + u_-,$$

where u_+ and u_- are boundary values of functions holomorphic on $\mathbb{D}$ and $\mathbb{D}^*$ respectively and $u_+(0) = 0$. Explicitly,

$$\begin{aligned} u_+(z) =& i\sum_{n=1}^{\infty} c_n z^{n+1}, \\ u_-(z) =& i\sum_{n=1}^{\infty} c_{-n} z^{1-n}. \end{aligned}$$

It is a difficult problem to characterize the Zygmund class in terms of the Fourier series (cf. Remark 2.4). On the other side, in terms of the Fourier series the almost complex structure J on $T_0^{\mathbb{R}} S^1\backslash\mathrm{Homeo}(S^1)$ is explicitly given by the classical conjugation operator

$$J\,\mathrm{u} = i\sum_{n\in\mathbb{Z}\backslash\{0\}} \mathrm{sgn}(n) c_n e^{in\theta}\frac{d}{d\theta} \quad \text{for} \quad \mathrm{u} = \sum_{n\in\mathbb{Z}\backslash\{0\}} c_n e^{in\theta}\frac{d}{d\theta}.$$

Remark 2.8. Note that our definition of the operator J differs by a negative sign from the definition in [Kir87, NV90] for the homogeneous space $S^1\backslash\operatorname{Diff}_+(S^1)$.

The holomorphic and anti-holomorphic tangent vectors at the origin are

$$\mathrm{v}=\frac{\mathrm{u}-iJ\,\mathrm{u}}{2}=\sum_{n=1}^{\infty}c_n e^{in\theta}\frac{d}{d\theta}\quad\text{and}\quad\bar{\mathrm{v}}=\frac{\mathrm{u}+iJ\,\mathrm{u}}{2}=\sum_{n=-\infty}^{-1}c_n e^{in\theta}\frac{d}{d\theta}.$$

For every smooth function $\mathcal{F}$ in a neighborhood of the origin in $\mathcal{T}(1)$ and $\mathrm{u}\in T_0^{\mathbb{R}}\,S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ set

$$\dot{\mathcal{F}}[\mathrm{u}]=\left.\frac{d}{dt}\right|_{t=0}\mathcal{F}(\gamma^t),$$

where γ_t is a curve in $S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ with the tangent vector u at the origin. Corresponding directional derivatives of $\mathcal{F}$ at the origin in $\mathcal{T}(1)$ in the holomorphic and anti-holomorphic directions v and $\bar{\mathrm{v}}$ are defined by
(2.15)

$$\partial\mathcal{F}(\mathrm{v})=\frac{1}{2}\left(\dot{\mathcal{F}}[\mathrm{u}]-i\,\dot{\mathcal{F}}[J\,\mathrm{u}]\right),\quad\text{and}\quad\bar{\partial}\mathcal{F}(\bar{\mathrm{v}})=\frac{1}{2}\left(\dot{\mathcal{F}}[\mathrm{u}]+i\,\dot{\mathcal{F}}[J\,\mathrm{u}]\right).$$

For $s\in\mathbb{R}$ let

$$\mathcal{H}^s(S^1)=\left\{\mathrm{u}=\sum_{n=-\infty}^{\infty}a_n e^{in\theta}\frac{d}{d\theta}:\sum_{n=-\infty}^{\infty}|n|^{2s}|a_n|^2<\infty\right\}$$

be the Sobolev space of complex-valued vector fields on S^1. The properties of tangent spaces $T_0S^1\backslash\mathrm{Homeo}_{qs}(S^1)$, $T_0\widetilde{\mathcal{D}}$ and $T_0\,\text{Möb}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)$, which will be used in Section 5, can be succinctly summarized as follows (see [Teo02] for details).

TS1. Under the $\mathbb{R}$-linear isomorphism $T_0^{\mathbb{R}}\,S^1\backslash\mathrm{Homeo}_{qs}(S^1)\xrightarrow{\sim}T_0\widetilde{\mathcal{D}}$

$$u(\theta)=\sum_{n\in\mathbb{Z}\backslash\{0\}}c_n e^{in\theta}\mapsto u_+(z)=i\sum_{n=1}^{\infty}c_n z^{n+1},$$

and $\dot{f}|_{\mathbb{D}}=u_+$, $\dot{g}_0|_{\mathbb{D}^*}=-u_-$, where

$$\dot{f}=\left.\frac{d}{dt}f^t\right|_{t=0},\quad\dot{g}=\left.\frac{d}{dt}g_t\right|_{t=0},$$

$\gamma_t=g_t^{-1}\circ f^t$ is a smooth curve in $\mathcal{T}(1)$ tangent to u at the origin, and $\dot{g}_0(z)=\dot{g}(z)-\dot{g}(\infty)z$.

TS2. Under the $\mathbb{R}$-linear isomorphism

$$T_0^{\mathbb{R}}\,\text{Möb}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)\xrightarrow{\sim}T_0T(1)\xrightarrow{D_0\beta}A_\infty(\mathbb{D})$$

$$u(\theta)=\sum_{n\in\mathbb{Z}\backslash\{-1,0,1\}}c_n e^{in\theta}\mapsto\frac{d^3u_+}{dz^3}(z)=i\sum_{n=2}^{\infty}(n^3-n)c_n z^{n-2}.$$

TS3. If

$$\phi(z)=\sum_{n=2}^{\infty}(n^3-n)a_n z^{n-2}\in A_\infty(\mathbb{D})$$

then

$$\sum_{n=2}^{\infty} n^{2s}|a_n|^2<\infty \quad \text{for all} \quad s<1.$$

TS4. $T_0^{\mathbb{R}}S^1\backslash\mathrm{Homeo}_{qs}(S^1)\subset\mathcal{H}^s(S^1)$ for all $s<1$.

2.3. **Teichmüller spaces and Teichmüller curves of Fuchsian groups.** Let Γ be a Fuchsian group, i.e., a discrete subgroup of $\mathrm{PSU}(1,1)$. Let

$$L^\infty(\mathbb{D}^*,\Gamma)=\left\{\mu\in L^\infty(\mathbb{D}^*):\mu\circ\gamma\frac{\overline{\gamma'}}{\gamma'}=\mu \quad \text{for all} \quad \gamma\in\Gamma\right\}$$

be a space of bounded Beltrami differentials for Γ and

$$L^\infty(\mathbb{D}^*,\Gamma)_1=L^\infty(\mathbb{D}^*)_1\cap L^\infty(\mathbb{D}^*,\Gamma)\ ,$$

be an open unit ball in $L^\infty(\mathbb{D}^*,\Gamma)$. The Teichmüller space of the Fuchsian group Γ is defined by

$$T(\Gamma)=L^\infty(\mathbb{D}^*,\Gamma)_1/\sim\ ,$$

where the equivalence relation is the same as the one used to define the universal Teichmüller space $T(1)$ in Section 2.1.1. The Teichmüller space $T(\Gamma)$ has a natural structure of a complex Banach manifold such that the tangent space at the origin of $T(\Gamma)$ is identified with the Banach space $\Omega^{-1,1}(\mathbb{D}^*,\Gamma)$ of bounded harmonic Beltrami differentials for Γ,

$$\Omega^{-1,1}(\mathbb{D}^*,\Gamma)=\Omega^{-1,1}(\mathbb{D}^*)\cap L^\infty(\mathbb{D}^*,\Gamma).$$

For every Fuchsian group Γ the canonical embedding $T(\Gamma)\hookrightarrow T(1)$ is holomorphic, so that the universal Teichmüller space $T(1)$ contains all Teichmüller spaces $T(\Gamma)$ as complex submanifolds. The universal Teichmüller space $T(1)$ is the Teichmüller space for the trivial Fuchsian group $\Gamma=\{1\}$.

The inverse image of $T(\Gamma)$ under the projection map $\mathcal{T}(1)\to T(\Gamma)$ is called the Bers fiber space $\mathcal{BF}(\Gamma)$. The quasi-Fuchsian group $\Gamma^\mu=w^\mu\circ\Gamma\circ(w^\mu)^{-1}$ acts on the fiber $w^\mu(\mathbb{D}^*)$ at the point $[\mu]\in T(\Gamma)$. The Teichmüller curve of the Fuchsian group Γ is the fiber space $\mathcal{T}(\Gamma)$ over $T(\Gamma)$ with the fiber $\Gamma^\mu\backslash w^\mu(\mathbb{D}^*)$ at the point $[\mu]\in T(\Gamma)$.

The domain $\mathbb{D}^*$ is a model of the hyperbolic plane $\mathbb{H}^2$. The hyperbolic (Poincaré) metric on $\mathbb{D}^*$ — a Hermitian metric of constant Gaussian curvature -1, is

$$ds^2=\rho(z)|dz|^2=\frac{4|dz|^2}{(1-|z|^2)^2}\,, \tag{2.16}$$

and the hyperbolic area 2-form is $\rho(z)\,d^2z$. The Fuchsian group Γ is of finite type (cofinite) if the corresponding Riemann surface — the orbifold $\Gamma\backslash\mathbb{D}^*$, has a finite hyperbolic area. In this case, the Teichmüller space $T(\Gamma)$ is a finite-dimensional complex manifold with a natural Hermitian metric, called

Weil-Petersson metric. It is defined as Petersson's inner product on tangent spaces $T_{[\mu]}T(\Gamma) \simeq \Omega^{-1,1}(\mathbb{D}^*, \Gamma_\mu)$, where $[\mu] \in T(\Gamma)$ and $\Gamma_\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$. For $\mu, \nu \in T_0T(\Gamma)$,

$$\langle \mu, \nu \rangle_{WP} = \iint\limits_{\mathbb{D}^*} \mu\bar{\nu}\rho(z)d^2z.$$

The Weil-Petersson metric on $T(\Gamma)$ is a Kähler metric.

The following result, due to Patterson [Pat75], will be used in Section 8. Here we present it in a convenient form as in [Teo02].

Lemma 2.9. *Let Γ be a cofinite Fuchsian group and $h \in L^\infty(\mathbb{D}^*, \rho(z)d^2z)$ be Γ–automorphic, i.e., $h \circ \gamma = h$ for all $\gamma \in \Gamma$. Then*

$$\iint\limits_{\Gamma\backslash\mathbb{D}^*} h(z)\rho(z)d^2z = \lim_{r\to 1^+} \frac{A(\Gamma\backslash\mathbb{D}^*)}{A(\mathbb{D}_r^*)} \iint\limits_{\mathbb{D}_r^*} h(z)\rho(z)d^2z,$$

where $\mathbb{D}_r^* = \{z \in \mathbb{D}^* : |z| \geq r\}$, $A(\Gamma\backslash\mathbb{D}^*)$ is the hyperbolic area of the Riemann surface $\Gamma\backslash\mathbb{D}^*$, and $A(\mathbb{D}_r^*)$ is the hyperbolic area of $\mathbb{D}_r^*$.

2.4. **Resolvent kernel.** Let

$$\Delta_0 = -\rho(z)^{-1}\partial_z\partial_{\bar{z}} \tag{2.17}$$

be the Laplace-Beltrami operator of the hyperbolic metric on $\mathbb{D}$, acting on functions. It is well-known (see, e.g., [Hej76, Lan85]) that the differential expression (2.17) defines unique positive, self-adjoint operator on the Hilbert space $L^2(\mathbb{D}, \rho(z)d^2z)$, which we still denote by Δ_0. Let

$$G = \tfrac{1}{2}\left(\Delta_0 + \tfrac{1}{2}\right)^{-1}$$

be (a one-half of) the resolvent of Δ_0 at the regular point $\lambda = -\frac{1}{2}$.

Remark 2.10. Note that the Laplace-Beltrami operator in [Hej76, Lan85] is $4\Delta_0$, so that the regular point $\lambda = -\frac{1}{2}$ for the operator Δ_0 corresponds to $\lambda = -2$ for the Laplace-Beltrami operator in [Hej76, Lan85].

The resolvent G is a bounded integral operator on $L^2(\mathbb{D}, \rho(z)d^2z)$ with the kernel

$$G(z, w) = \frac{2u+1}{2\pi}\log\frac{u+1}{u} - \frac{1}{\pi}, \tag{2.18}$$

where $u(z, w)$ is a point-pair invariant on $\mathbb{D}$,

$$u(z, w) = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}.$$

The resolvent kernel $G(z, w)$ has the following properties (see, e.g., [Hej76] and [Lan85, Sect. XIV.3]).

RK1. G is symmetric, $G(z, w) = G(w, z)$, and is a point-pair invariant,

$$G(\gamma z, \gamma w) = G(z, w) \text{ for all } \gamma \in \mathrm{PSU}(1, 1).$$

RK2. $G(z, w)$ is positive for all $z, w \in \mathbb{D}$.

RK3. If $g \in BC^{\infty}(\mathbb{D})$ — the space of smooth bounded functions on $\mathbb{D}$, then the integral

$$f(z) = \iint\limits_{\mathbb{D}} G(z, w) g(w) \rho(w) d^2 w$$

is absolutely convergent for all $z \in \mathbb{D}$ and $f = G(g) \in BC^{\infty}(\mathbb{D})$ satisfies the differential equation

$$2\left(\Delta_0 + \tfrac{1}{2}\right)(f) = g.$$

Conversely, if $f \in BC^{\infty}(\mathbb{D})$ and $g = 2\left(\Delta_0 + \frac{1}{2}\right)(f) \in BC^{\infty}(\mathbb{D})$, then $f = G(g)$.

RK4. For all $z \in \mathbb{D}$,

$$\iint\limits_{\mathbb{D}} G(z, w) \rho(w) d^2 w = 1.$$

The last property immediately follows from **RK3** since

$$2\left(\Delta_0 + \tfrac{1}{2}\right)(1) = 1,$$

where 1 is a constant function equal to 1 on $\mathbb{D}$.

The resolvent kernel G of the Laplace-Beltrami operator on $\mathbb{D}^*$ is given by the same formula (2.18) and satisfies the properties **RK1** – **RK4**.

When Γ is a cofinite Fuchsian group, we denote by G_Γ the one-half of the resolvent of the Laplace-Beltrami operator on the Riemann surface $\Gamma \backslash \mathbb{D}$ at $\lambda = -\frac{1}{2}$. It is a bounded integral operator on $L^2(\Gamma \backslash \mathbb{D}, \rho(z) d^2 z)$ with the kernel

$$G_\Gamma(z, w) = \sum_{\gamma \in \Gamma} G(z, \gamma w),\ z, w \in \mathbb{D}, \tag{2.19}$$

and it enjoys all the properties **RK1**-**RK4**. The corresponding resolvent kernel on $\Gamma \backslash \mathbb{D}^*$ is given by the same formula with $z, w \in \mathbb{D}^*$.

Remark 2.11. The operator G_Γ plays a prominent role in the Weil-Petersson geometry of the finite-dimensional Teichmüller space $T(\Gamma)$ [Wol86].

2.5. **Variational formulas.** Here we collect necessary variational formulas. To simplify the computations in the following sections, we will use different realizations of the hyperbolic plane $\mathbb{H}^2$, given either by the unit disk $\mathbb{D}$ or its exterior $\mathbb{D}^*$, or by the upper half-plane $\mathbb{U}$.

Let l and m be integers and Γ a Fuchsian group (we will be primarily interested in the cases when $\Gamma = \{1\}$, i.e., is a trivial group, and when Γ is a cofinite Fuchsian group). Using the model $\mathbb{H}^2 \simeq \mathbb{D}$, tensor of type (l, m) for Γ is a C^{∞}-function ω on $\mathbb{D}$ satisfying

$$\omega(\gamma z) \gamma'(z)^l \overline{\gamma'(z)}^m = \gamma(z) \text{ for all } \gamma \in \Gamma.$$

Let ω^{ε} be a smooth family of tensors of type (l,m) for $\Gamma_{\varepsilon\mu}=w_{\varepsilon\mu}\circ\Gamma\circ w_{\varepsilon\mu}^{-1}$, where $\mu\in\Omega^{-1,1}(\mathbb{D},\Gamma)$ and $\varepsilon\in\mathbb{C}$ is sufficiently small. Set

$$(w_{\varepsilon\mu})^{*}(\omega^{\varepsilon})=\omega^{\varepsilon}\circ w_{\varepsilon\mu}((w_{\varepsilon\mu})_{z})^{l}((\overline{w}_{\varepsilon\mu})_{\bar{z}})^{m},$$

which is a tensor of type (l,m) for Γ — a pull-back of the tensor ω^{ε} by $w_{\varepsilon\mu}$. Lie derivatives of the family ω^{ε} along vector fields $\partial/\partial\varepsilon_{\mu}$ and $\partial/\partial\bar{\varepsilon}_{\mu}$ are defined in the standard way,

$$L_{\mu}\omega=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}(w_{\varepsilon\mu})^{*}(\omega^{\varepsilon})\quad\text{and}\quad L_{\bar{\mu}}\omega=\frac{\partial}{\partial\bar{\varepsilon}}\Big|_{\varepsilon=0}(w_{\varepsilon\mu})^{*}(\omega^{\varepsilon}).$$

When ω is a function on $T(\Gamma)$ — a tensor of type $(0,0)$, the Lie derivatives reduce to directional derivatives

$$L_{\mu}\omega=\partial\omega(\mu)\quad\text{and}\quad L_{\bar{\mu}}\omega=\bar{\partial}\omega(\bar{\mu})$$

— the evaluation of 1-forms $\partial\omega$ and $\bar{\partial}\omega$ on the holomorphic and antiholomorphic tangent vectors μ and $\bar{\mu}$ to $T(\Gamma)$ at the origin. Corresponding real vector fields $\frac{\partial}{\partial t_{\mu}}$ are defined by

$$\frac{\partial}{\partial t_{\mu}}=\frac{\partial}{\partial\varepsilon_{\mu}}+\frac{\partial}{\partial\bar{\varepsilon}_{\mu}},$$

so that

$$\frac{\partial}{\partial\varepsilon_{\mu}}=\frac{1}{2}\left(\frac{\partial}{\partial t_{\mu}}-i\frac{\partial}{\partial t_{i\mu}}\right)\quad\text{and}\quad\frac{\partial}{\partial\bar{\varepsilon}_{\mu}}=\frac{1}{2}\left(\frac{\partial}{\partial t_{\mu}}+i\frac{\partial}{\partial t_{i\mu}}\right).$$

For the model $\mathbb{H}^2\simeq\mathbb{U}$ we have

$$\frac{\partial}{\partial\varepsilon}w_{\varepsilon\mu}(z)=-\frac{1}{\pi}\iint\limits_{\mathbb{U}}R\left(w_{\varepsilon\mu}(z),w_{\varepsilon\mu}(u)\right)\mu(u)(w_{\varepsilon\mu})_{u}^{2}(u)d^{2}u,\tag{2.20}$$

$$\frac{\partial}{\partial\bar{\varepsilon}}w_{\varepsilon\mu}(z)=-\frac{1}{\pi}\iint\limits_{\mathbb{U}}R\left(w_{\varepsilon\mu}(z),\overline{w_{\varepsilon\mu}(u)}\right)\overline{\mu(u)(w_{\varepsilon\mu})_{u}^{2}(u)}d^{2}u,$$

where the q.c. mapping $w_{\varepsilon\mu}$ is normalized by fixing $0,1,\infty$ and the kernel R is

$$R(z,u)=\frac{z(z-1)}{(u-z)u(u-1)}=\frac{1}{u-z}+\frac{z-1}{u}-\frac{z}{u-1}.$$

Setting

$$F[\mu]=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}w_{\varepsilon\mu}\quad\text{and}\quad\Phi[\mu]=\frac{\partial}{\partial\bar{\varepsilon}}\Big|_{\varepsilon=0}w_{\varepsilon\mu}.$$

we get from (2.20)

$$F[\mu](z)=-\frac{1}{\pi}\iint\limits_{\mathbb{U}}R(z,u)\mu(u)\,d^{2}u,\tag{2.21}$$

$$\Phi[\mu](z)=-\frac{1}{\pi}\iint\limits_{\mathbb{U}}R(z,\bar{u})\overline{\mu(u)}\,d^{2}u.$$

The function $\Phi[\mu](z)$ is holomorphic on $\mathbb{U}$ and satisfies

$$\Phi[\mu]_{zzz}(z) = -\frac{6}{\pi} \iint\limits_{\mathbb{U}} \frac{\overline{\mu(u)}}{(\bar{u} - z)^4} \, d^2u.$$

As it follows from (2.7), the projection $P : L^\infty(\mathbb{U}) \to \Omega^{-1,1}(\mathbb{U})$ is given by

$$(P\mu)(z) = -\frac{3(z - \bar{z})^2}{\pi} \iint\limits_{\mathbb{U}} \frac{\mu(u)}{(u - \bar{z})^4} d^2u. \tag{2.22}$$

Equivalently, for $\mu(z) = \frac{(z-\bar{z})^2}{2}\overline{\phi(z)}$ with $\phi \in A_\infty(\mathbb{U})$, $\Phi[\mu]_{zzz} = \phi$ on $\mathbb{U}$. The function $F[\mu]$ satisfies $F[\mu]_{\bar{z}} = \mu$ on $\mathbb{U}$, and is holomorphic on the lower half-plane $\overline{\mathbb{U}}$.

Lemma 2.12. *For $\mu \in \Omega^{-1,1}(\mathbb{U})$ and $z \in \mathbb{U}$,*

$$\lim_{\varepsilon \to 0} \iint\limits_{\mathbb{U}(z,\varepsilon)} \frac{\mu(u)}{(u - z)^4} d^2u = \lim_{\varepsilon \to 0} \iint\limits_{\mathbb{U}(z,\varepsilon)} \frac{\mu(u)}{(u - z)^5} d^2u = 0,$$

where $\mathbb{U}(z, \varepsilon) = \mathbb{U} \setminus \{u \in \mathbb{U} : |u - z| < \varepsilon\}$.

Proof. The proof of the first formula essentially follows the classical Ahlfors' proof in [Ahl87, Lemma 2 in Sect. VI D] by using $\mu(u) = \frac{(u-\bar{u})^2}{2}\overline{\phi(u)}$ with $\phi \in A_\infty(\mathbb{U})$, the identity

$$\frac{(u - \bar{u})^2}{(u - z)^4} = \frac{\partial}{\partial u}\left(-\frac{1}{u - z} + \frac{\bar{u} - z}{(u - z)^2} - \frac{1}{3}\frac{(\bar{u} - z)^2}{(u - z)^3}\right),$$

and the Stokes' theorem. The second formula is proved similarly. □

Another classical result of Ahlfors [Ahl61] is the following.

Lemma 2.13. *For $\mu \in \Omega^{-1,1}(\mathbb{U})$ and $z \in \mathbb{U}$,*

$$F[\mu](z) = \frac{(z - \bar{z})^2}{2}\overline{\Phi''(z)} + (z - \bar{z})\overline{\Phi'(z)} + \overline{\Phi(z)},$$

where $\Phi(z) = \Phi[\mu](z)$.

Remark 2.14. It follows from Lemma 2.13 that $F[\mu]_{zzz} = 0$ for $\mu \in \Omega^{-1,1}(\mathbb{U})$, in agreement with Lemma 2.12.

Corollary 2.15. *For $\mu \in \Omega^{-1,1}(\mathbb{U})$ and $z \in \mathbb{U}$,*

$$\iint\limits_{\mathbb{U}} \frac{\mu(u)}{(u - z)(u - \bar{z})^3} d^2u = 0.$$

Proof. Using (2.21), we have

$$\begin{aligned} &F[\mu](z) - \frac{(z - \bar{z})^2}{2}\overline{\Phi''(z)} - (z - \bar{z})\overline{\Phi'(z)} - \overline{\Phi(z)} \\ &= -\frac{1}{\pi} \iint\limits_{\mathbb{U}} \mu(u) \frac{(z - \bar{z})^3}{(u - z)(u - \bar{z})^3} d^2w. \end{aligned}$$

□

For $\mu \in L^\infty(\mathbb{U})_1$ set

$$K_\mu(u,v) = \frac{(w_\mu)_u(u)(w_\mu)_v(v)}{(w_\mu(u)-w_\mu(v))^2} \quad \text{and} \quad K_\mu(u,\bar{v}) = \frac{(w_\mu)_u(u)\overline{(w_\mu)_v(v)}}{(w_\mu(u)-\overline{w_\mu(v)})^2}.$$

We have from (2.20) the following formulas [Ahl62]

$$\begin{aligned} \frac{\partial}{\partial\varepsilon} K_{\varepsilon\mu}(z,u) &= -\frac{1}{\pi}\iint\limits_{\mathbb{U}} \mu(v) K_{\varepsilon\mu}(z,v)K_{\varepsilon\mu}(v,u)\, d^2v, \\ \frac{\partial}{\partial\bar\varepsilon} K_{\varepsilon\mu}(z,u) &= -\frac{1}{\pi}\iint\limits_{\mathbb{U}} \overline{\mu(v)} K_{\varepsilon\mu}(z,\bar v)K_{\varepsilon\mu}(\bar v,u)\, d^2v, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \frac{\partial}{\partial\varepsilon} K_{\varepsilon\mu}(z,\bar u) &= -\frac{1}{\pi}\iint\limits_{\mathbb{U}} \mu(v) K_{\varepsilon\mu}(z,v)K_{\varepsilon\mu}(v,\bar u)\, d^2v, \\ \frac{\partial}{\partial\bar\varepsilon} K_{\varepsilon\mu}(z,\bar u) &= -\frac{1}{\pi}\iint\limits_{\mathbb{U}} \overline{\mu(v)} K_{\varepsilon\mu}(z,\bar v)K_{\varepsilon\mu}(\bar v,\bar u)\, d^2v, \end{aligned} \tag{2.24}$$

where the integrals are understood in the principal value sense.

For the model $\mathbb{H}^2 \simeq \mathbb{D}$ the q.c. mapping w_μ is normalized by fixing $-1, -i, 1$. The kernel R is given by

$$R(z,u) = \frac{(z+1)(z+i)(z-1)}{(u-z)(u+1)(u+i)(u-1)},$$

and formulas similar to (2.21) hold for F and Φ. In particular, let f be a q.c. mapping such that $f|_{\mathbb{D}} \in \mathcal{D}$, and let μ be a Beltrami differential supported on the quasi-disk $\Omega^* = f(\mathbb{D}^*)$. Let $v_{t\mu}$ be the solution on $\mathbb{C}$ of the Beltrami equation

$$(v_{t\mu})_{\bar z} = t\mu (v_{t\mu})_z,$$

satisfying $v_{t\mu}(0)=0, v'_{t\mu}(0)=1$ and $v''_{t\mu}(0)=0$. Then

$$\dot v = \left.\frac{d}{dt}\right|_{t=0} v_{t\mu}$$

is a holomorphic function on $\Omega = f(\mathbb{D})$ and

$$\dot v_{zzz}(z) = -\frac{6}{\pi}\iint\limits_{\Omega^*} \frac{\mu(u)}{(u-z)^4} d^2u. \tag{2.25}$$

3. $T(1)$ as a Hilbert manifold

Here we endow $T(1)$ with a structure of a complex manifold modeled on the separable Hilbert space

$$A_2(\mathbb{D}) = \left\{ \phi \text{ holomorphic on } \mathbb{D} : \|\phi\|_2^2 = \iint\limits_{\mathbb{D}} |\phi|^2 \rho^{-1}(z) d^2z < \infty \right\}$$

of holomorphic functions on $\mathbb{D}$. In the corresponding topology, the universal Teichmüller space $T(1)$ is a disjoint union of uncountably many components on which the right translations act transitively.

3.1. **Hilbert space structure on tangent spaces.**

Let

$$A_2(\mathbb{D}^*) = \left\{ \phi \text{ holomorphic on } \mathbb{D}^* : \|\phi\|_2^2 = \iint\limits_{\mathbb{D}^*} |\phi|^2 \rho^{-1}(z) d^2z < \infty \right\}$$

be the Hilbert space of holomorphic functions on $\mathbb{D}^*$.

Lemma 3.1. *The vector spaces $A_2(\mathbb{D})$ and $A_2(\mathbb{D}^*)$ are subspaces of $A_\infty(\mathbb{D})$ and $A_\infty(\mathbb{D}^*)$ respectively. The natural inclusion maps $A_2(\mathbb{D}) \hookrightarrow A_\infty(\mathbb{D})$ and $A_2(\mathbb{D}^*) \hookrightarrow A_\infty(\mathbb{D}^*)$ are bounded linear mappings of Banach spaces.*

Proof. It is sufficient to consider only the spaces of holomorphic functions on $\mathbb{D}$. For every $\phi \in A_2(\mathbb{D})$ let $\phi = \sum_{n=2}^{\infty} (n^3 - n) a_n z^{n-2}$ be the power series expansion. Then

$$\|\phi\|_2^2 = \iint\limits_{\mathbb{D}} |\phi|^2 \rho^{-1} d^2z = \frac{\pi}{2} \sum_{n=2}^{\infty} (n^3 - n)|a_n|^2,$$

and by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\phi(z)| &= \left| \sum_{n=2}^{\infty} (n^3 - n) a_n z^{n-2} \right| \\ &\leq \left(\sum_{n=2}^{\infty} (n^3 - n)|a_n|^2 \right)^{1/2} \left(\sum_{n=2}^{\infty} (n^3 - n)|z|^{2n-4} \right)^{1/2} \end{aligned}$$

for every $z \in \mathbb{D}$. Since

$$\sum_{n=2}^{\infty} (n^3 - n)|z|^{2n-4} = \frac{6}{(1 - |z|^2)^4},$$

we have

$$\|\phi\|_\infty = \sup_{z \in \mathbb{D}} \left| (1 - |z|^2)^2 \phi(z) \right| \leq \sqrt{\frac{12}{\pi}} \|\phi\|_2.$$

□

Similarly, let

$$H^{-1,1}(\mathbb{D}) = \left\{ \mu = \rho^{-1}\bar{\phi},\ \phi \text{ holomorphic on } \mathbb{D} : \|\mu\|_2^2 = \iint_{\mathbb{D}} |\mu|^2 \rho(z) d^2z < \infty \right\}$$

and

$$H^{-1,1}(\mathbb{D}^*) = \left\{ \mu = \rho^{-1}\bar{\phi},\ \phi \text{ holomorphic on } \mathbb{D}^* : \|\mu\|_2^2 = \iint_{\mathbb{D}^*} |\mu|^2 \rho(z) d^2z < \infty \right\}$$

be the Hilbert spaces of harmonic Beltrami differentials on $\mathbb{D}$ and $\mathbb{D}^*$ respectively. It follows from Lemma 3.1 that the natural inclusion maps $H^{-1,1}(\mathbb{D}) \hookrightarrow \Omega^{-1,1}(\mathbb{D})$ and $H^{-1,1}(\mathbb{D}^*) \hookrightarrow \Omega^{-1,1}(\mathbb{D}^*)$ are bounded and under the Bers embedding $D_0\beta : H^{-1,1}(\mathbb{D}^*) \xrightarrow{\sim} A_2(\mathbb{D}) \hookrightarrow A_\infty(\mathbb{D})$.

Remark 3.2. It follows from the proof of Lemma 3.1 that every $\mu \in H^{-1,1}(\mathbb{D})$ (respectively in $H^{-1,1}(\mathbb{D}^*)$) satisfies

$$\lim_{|z|\to 1} \mu(z) = 0.$$

Indeed, for given $\varepsilon > 0$ let N be such that

$$\sum_{n=N}^{\infty} (n^3 - n)|a_n|^2 < \varepsilon.$$

Then

$$\begin{aligned} |\mu(z)| \leq & (1-|z|^2)^2 \left| \sum_{n=2}^{N-1} (n^3-n) a_n z^{n-2} \right| \\ & + (1-|z|^2)^2 \left(\sum_{n=N}^{\infty} (n^3-n)|a_n|^2 \right)^{1/2} \left(\sum_{n=2}^{\infty} (n^3-n)|z|^{2n-4} \right)^{1/2} \\ \leq & (1-|z|^2)^2 \left| \sum_{n=2}^{N-1} (n^3-n) a_n z^{n-2} \right| + \sqrt{6\varepsilon}, \end{aligned}$$

so that

$$\limsup_{|z|\to 1} |\mu(z)| \leq \sqrt{6\varepsilon}.$$

Since ε is arbitrary this proves the assertion.

For every $[\mu] \in T(1)$ let $D_0 R_{[\mu]}\left(H^{-1,1}(\mathbb{D}^*)\right)$ be the subspace of the tangent space $T_{[\mu]}T(1) = D_0 R_{[\mu]}\left(\Omega^{-1,1}(\mathbb{D}^*)\right)$ with a Hilbert space structure isomorphic to $H^{-1,1}(\mathbb{D}^*)$. Let $\mathfrak{D}_T$ be the distribution on $T(1)$, defined by the assignment

$$T(1) \ni [\mu] \mapsto D_0 R_{[\mu]}\left(H^{-1,1}(\mathbb{D}^*)\right) \subset T_{[\mu]}T(1).$$

Similarly, let $\mathfrak{D}_A$ be the distribution on $A_\infty(\mathbb{D})$, defined by

$$A_\infty(\mathbb{D}) \ni \phi \mapsto A_2(\mathbb{D}) \subset T_\phi A_\infty(\mathbb{D}) \simeq A_\infty(\mathbb{D}).$$

The next statement asserts that under the Bers embedding $\beta : T(1) \to A_\infty(\mathbb{D})$ the distribution $\mathfrak{D}_T$ is isomorphic to the restriction of the distribution $\mathfrak{D}_A$ to $\beta(T(1))$.

Theorem 3.3. *For every $[\mu] \in T(1)$ the linear mapping*

$$D_0\left(\beta \circ R_{[\mu]}\right) : H^{-1,1}(\mathbb{D}^*) \to A_2(\mathbb{D})$$

is a topological isomorphism.

Proof. Let $\nu \in H^{-1,1}(\mathbb{D}^*)$. Set $w_t = w_{t\nu * \mu} = w_{t\nu} \circ w_\mu$ and consider the factorization $\gamma_t = \alpha_t \circ w_t = g_t^{-1} \circ f^t$ associated with the q.c. mapping w_t by (2.10). Let $v_t = f^t \circ f^{-1}$, where $\gamma = \alpha \circ w_\mu = g^{-1} \circ f$ is the factorization for w_μ, and set $\Omega = f(\mathbb{D}) = g(\mathbb{D})$, $\Omega^* = f(\mathbb{D}^*) = g(\mathbb{D}^*)$. Since

$$\beta([t\nu * \mu]) = \mathcal{S}(f^t) = \mathcal{S}(v_t) \circ f f_z^2 + \mathcal{S}(f),$$

we have

$$D_0\left(\beta \circ R_{[\mu]}\right)(\nu) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{S}(f^t) = \dot{v}_{zzz} \circ f f_z^2, \quad \text{where} \quad \dot{v} = \left.\frac{d}{dt}\right|_{t=0} v_t.$$

The q.c. mapping v_t is holomorphic on Ω and satisfies $v_t \circ \tilde{g} = g_t \circ \alpha_t \circ w_{t\nu}$, where $\tilde{g} = g \circ \alpha$. Since g_t and g are holomorphic on $\mathbb{D}^*$, the Beltrami differential of v_t is given by

$$t\tilde{\nu}(z) = \begin{cases} 0, & z \in \Omega, \\ t(\nu \circ \tilde{g}^{-1})(z) \dfrac{\overline{\tilde{g}_z^{-1}(z)}}{\tilde{g}_z^{-1}(z)}, & z \in \Omega^*. \end{cases}$$

It follows from (2.25) that

(3.1)

$$D_0\left(\beta \circ R_{[\mu]}\right)(\nu)\left(f^{-1}(z)\right)\left(f_z^{-1}(z)\right)^2 = \dot{v}_{zzz}(z) = -\frac{6}{\pi} \iint\limits_{\Omega^*} \frac{\tilde{\nu}(u)}{(u-z)^4} d^2u.$$

Let $\rho_1(z) = (\rho \circ f^{-1})(z)|f_z^{-1}(z)|^2$ and $\rho_2(z) = (\rho \circ g^{-1})(z)|g_z^{-1}(z)|^2$ be the hyperbolic metric densities on the domains Ω and Ω^* respectively. Classical inequalities (see e.g., [Leh87, Nag88])

$$\frac{1}{16} \le \eta_i^2(z)\rho_i(z) \le 1, \qquad i = 1, 2,$$

where $\eta_1(z)$ and $\eta_2(z)$ stand, respectively, for the distances of $z \in \Omega$ and $z \in \Omega^*$ to the quasi-circle $f(S^1)$, yield the following estimates (cf. [Nag88, Sect. 3.4.5])

$$\iint\limits_{\Omega} \frac{d^2z}{|u-z|^4} \le \iint\limits_{|z-u| \ge \eta_2(u)} \frac{d^2z}{|u-z|^4} = \frac{\pi}{2\eta_2(u)^2} \le 8\pi\rho_2(u), \quad u \in \Omega^*,$$

and

$$\iint_{\Omega^*} \frac{d^2u}{|u-z|^4} \leq 8\pi\rho_1(z), \quad z \in \Omega.$$

From here it follows

$$\|D_0\left(\beta \circ R_{[\mu]}\right)(\nu)\|_2^2 = \iint_{\mathbb{D}} \left|D_0\left(\beta \circ R_{[\mu]}\right)(\nu)\right|^2 \rho^{-1} d^2z = \iint_{\Omega} \left|\dot{v}_{zzz}\right|^2 \rho_1^{-1} d^2z$$

$$\leq \frac{6^2}{\pi^2} \iint_{\Omega} \rho_1(z)^{-1} \iint_{\Omega^*} \frac{d^2v}{|v-z|^4} \iint_{\Omega^*} \frac{|\tilde{\nu}(u)|^2 d^2u}{|u-z|^4}\, d^2z$$

$$\leq \frac{6^2 \cdot 8}{\pi} \iint_{\Omega} \iint_{\Omega^*} \frac{|\tilde{\nu}(u)|^2 d^2u}{|u-z|^4}\, d^2z \leq 6^2 \cdot 8^2 \iint_{\Omega^*} |\tilde{\nu}(u)|^2 \rho_2(u) d^2u$$

$$= 6^2 \cdot 8^2 \iint_{\mathbb{D}^*} |\nu|^2 \rho(u) d^2u = 2304\|\nu\|_2^2.$$

To prove that the mapping $D_0\left(\beta \circ R_{[\mu]}\right)$ is onto, we adapt to our case Bers' arguments, as presented in [Nag88, Sect. 3.5]. For $\phi \in A_2(\mathbb{D})$ set $q = (\phi \circ f^{-1})(f_z^{-1})^2$, and choose μ in the equivalence class of $[\mu] \in T(1)$ to be the conformally natural extension of $(g^{-1} \circ f)|_{S^1}$, constructed by Douady and Earle [DE86]. Let h be the corresponding quasiconformal reflection [EN88] on $\mathbb{C}$ which fixes the quasi-circle $f(S^1)$. According to the Bers reproducing formula [Ber66],

$$q(z) = -\frac{3}{\pi} \iint_{\Omega^*} \frac{q(h(u))(u-h(u))^2 h_{\bar{u}}(u)}{(u-z)^4} d^2u. \tag{3.2}$$

Analogously to $L^\infty(\mathbb{D}^*)$ and $\Omega^{-1,1}(\mathbb{D}^*)$ consider the Banach spaces $L^\infty(\Omega^*)$ and

$$\Omega^{-1,1}(\Omega^*) = \left\{\mu \in L^\infty(\Omega^*) : \mu = 4\rho_2^{-1}\bar{q},\ q \text{ is holomorphic on } \Omega^*\right\},$$

and denote by $\tilde{P}$ the corresponding projection $\tilde{P} : L^\infty(\Omega^*) \to \Omega^{-1,1}(\Omega^*)$. The mapping

$$(\tilde{g}^*)^{-1}(\mu) = \mu \circ \tilde{g}^{-1} \frac{\overline{(\tilde{g}^{-1})'}}{(\tilde{g}^{-1})'}$$

establishes the isomorphisms $L^\infty(\mathbb{D}^*) \simeq L^\infty(\Omega^*)$ and $\Omega^{-1,1}(\mathbb{D}^*) \simeq \Omega^{-1,1}(\Omega^*)$, and $\tilde{P} = (\tilde{g}^*)^{-1} \circ P \circ \tilde{g}^*$. Define $\nu \in \Omega^{-1,1}(\mathbb{D}^*)$ by

$$(\tilde{g}^*)^{-1}(\nu)(z) = \tilde{P}\left(\frac{1}{2}(q(h(z))(z-h(z))^2 h_{\bar{z}}(z)\right) \in \Omega^{-1,1}(\Omega^*).$$

The comparison between (3.2) and (3.1) shows that

$$D_0\left(\beta \circ R_{[\mu]}\right)(\nu) = \phi.$$

To prove that $\nu \in H^{-1,1}(\mathbb{D}^*)$ we use the Earle-Nag [EN88] estimate,

$$\frac{1}{C} \leq |z - h(z)|^4 \rho_1(h(z))\rho_2(z) \leq C, \quad z \in \Omega^*, \tag{3.3}$$

where the constant C depends only on $\|\mu\|_\infty$. Since operator $\tilde{P}$ also gives the orthogonal projection of $L^2(\Omega^*, \rho_2(z)\, d^2z)$ onto $H^{-1,1}(\Omega^*)$, we get by the Earle-Nag inequality

$$\begin{aligned}\iint\limits_{\mathbb{D}^*} |\nu|^2\rho(z)d^2z &= \iint\limits_{\Omega^*} |(\tilde{g}^*)^{-1}(\nu)(z)|^2\rho_2(z)d^2z \\ &\leq \frac{1}{4}\iint\limits_{\Omega^*} |q(h(z))(z-h(z))^2 h_{\bar{z}}(z)|^2\rho_2(z)d^2z \\ &\leq \frac{C}{4}\iint\limits_{\Omega^*} |q(h(z))h_{\bar{z}}(z)|^2\rho_1^{-1}(h(z))d^2z.\end{aligned}$$

Since h is sense reversing, for

$$\kappa = \frac{h_z^{-1}}{h_{\bar{z}}^{-1}}$$

we have $\|\kappa\|_\infty < 1$. Now

$$\begin{aligned}&\iint\limits_{\Omega^*} |q(h(z))h_{\bar{z}}(z)|^2\rho_1^{-1}(h(z))d^2z \\ =&\iint\limits_{\Omega} |q(z)|^2\rho_1^{-1}(z)|(h_{\bar{z}}\circ h^{-1})(z)h_{\bar{z}}^{-1}(z)|^2(1-|\kappa|^2)d^2z \\ =&\iint\limits_{\Omega} \frac{|q(z)|^2}{1-|\kappa|^2}\rho_1^{-1}(z)d^2z \\ \leq& C_1\iint\limits_{\Omega} |q(z)|^2\rho_1(z)^{-1}d^2z = C_1\|\phi\|_2^2 < \infty,\end{aligned}$$

so that $\|\nu\|_2 \leq C_2\|\phi\|_2$. This also proves that the inverse map to $D_0\left(\beta\circ R_{[\mu]}\right)$ is bounded, so that $D_0\left(\beta\circ R_{[\mu]}\right)$ is a topological isomorphism. □

Remark 3.4. It follows from the proof of the first part of Theorem 3.3 that $D_0\left(\beta\circ R_{[\mu]}\right) = D_0\left(\beta\circ\Phi\circ R_\mu\right)$ extends to a bounded linear operator in $L^2(\mathbb{D}^*, \rho(z)d^2z)$ and the estimate

$$\|D_0\left(\beta\circ\Phi\circ R_\mu\right)(\nu)\|_2 = \|D_\mu\left(\beta\circ\Phi\right)\left(D_0R_\mu(\nu)\right)\|_2 \leq 48\|\nu\|_2$$

holds for all $\nu \in L^2(\mathbb{D}^*, \rho(z)d^2z)$ and $\mu \in L^\infty(\mathbb{D}^*)_1$.

3.2. **The L^2-estimates.** The lemmas below are needed for the rigorous definition of a complex Hilbert manifold structure on $T(1)$.

Lemma 3.5. *For every $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for all $\mu \in \Omega^{-1,1}(\mathbb{D}^*)$ with $\|\mu\|_\infty < \delta$,*

$$\left| \frac{|(w_\mu)_z(z)|^2}{(1-|w_\mu(z)|^2)^2} - \frac{1}{(1-|z|^2)^2} \right| < \frac{\varepsilon}{(1-|z|^2)^2}$$

for all $z \in \mathbb{D} \cup \mathbb{D}^*$. The same inequality holds for $w_{\mu^{-1}} = w_\mu^{-1}$.

Proof. Using the isomorphism $\Omega^{-1,1}(\mathbb{D}^*) \xrightarrow{\sim} \Omega^{-1,1}(\mathbb{D})$, given by the reflection (2.1) and the property (2.2), it is sufficient to prove the estimate for $z \in \mathbb{D}$. Since $\gamma_\mu = \alpha \circ w_\mu$, where $\alpha \in \mathrm{PSU}(1,1)$, the estimate holds for w_μ if and only if it holds for γ_μ. By Lemma 2.5 γ_μ fixes 0 and ∞, and by the result of Ahlfors and Bers in [AB60] (see also the remark of Bers in [Ber73]) the functional

$$\mathcal{L}^\infty(\mathbb{D}^*)_1 \ni \mu \mapsto (\gamma_\mu)_z(0) \in \mathbb{C}$$

is real-analytic at $\mu = 0$. In particular, for every $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for all $\mu \in \Omega^{-1,1}(\mathbb{D}^*)_1$ with $\|\mu\|_\infty < \delta$,

$$\left| |(\gamma_\mu)_z(0)|^2 - 1 \right| < \varepsilon.$$

Let $\tilde{\mu} = \mu \circ \sigma_z$ and $\gamma_{\tilde{\mu}} = \tilde{\sigma}_z \circ \gamma_\mu \circ \sigma_z$, where $\sigma_z(w) = \frac{w+z}{1+\bar{z}w}$ and $\tilde{\sigma}_z \in \mathrm{PSU}(1,1)$. Since $\tilde{\mu} \in \Omega^{-1,1}(\mathbb{D}^*)_1$, it follows from Lemma 2.5 that also $\gamma_{\tilde{\mu}}(0) = 0$. Therefore $\tilde{\sigma}_z(\gamma_\mu(z)) = 0$, and one obtains

$$\frac{|(\gamma_\mu)_z(z)|^2}{(1-|\gamma_\mu(z)|^2)^2}(1-|z|^2)^2 = |(\gamma_{\tilde{\mu}})_z(0)|^2 .$$

Since $\|\tilde{\mu}\|_\infty = \|\mu\|_\infty$, the assertion follows. Since for $\mu \in \Omega^{-1,1}(\mathbb{D}^*)_1$, $\mu^{-1} \in \mathcal{L}^\infty(\mathbb{D}^*)_1$ and $\|\mu^{-1}\|_\infty = \|\mu\|_\infty$, the assertion also holds for $w_{\mu^{-1}}$. ☐

Corollary 3.6. *Let $\mu \in \Omega^{-1,1}(\mathbb{D}^*)$, $\|\mu\|_\infty < \delta$, where δ corresponds to $\varepsilon = 1$ in the previous lemma. Then for every $\lambda \in L^\infty(\mathbb{D}^*)_1$ the linear mapping $D_\lambda R_\mu$ extends to an invertible bounded linear operator on the Hilbert space $L^2(\mathbb{D}^*, \rho(z)d^2z)$. Moreover,*

$$\left\| D_\lambda R_\mu(\nu) \right\|_2 \le \frac{\sqrt{2}}{(1-\|\mu\|_\infty)^2} \|\nu\|_2,$$

for all $\nu \in L^2(\mathbb{D}^*, \rho(z)d^2z)$ and $\lambda \in L^\infty(\mathbb{D}^*)_1$, and the same inequality holds for $D_\lambda R_{\mu^{-1}}$.

Proof. Since

$$D_\lambda R_\mu(\nu) = \frac{(1-|\mu|^2)\, \nu \circ w_\mu}{\left(1 + \overline{\mu} \lambda \circ w_\mu \frac{\overline{(w_\mu)_z}}{(w_\mu)_z}\right)^2} \frac{\overline{(w_\mu)_z}}{(w_\mu)_z},$$

and

$$\left\| \frac{1-|\mu|^2}{\left(1+\overline{\mu}\lambda\circ w_\mu \frac{\overline{(w_\mu)_z}}{(w_\mu)_z}\right)^2} \right\|_\infty \leq \frac{1}{(1-\|\mu\|_\infty)^2}$$

for all $\lambda \in L^\infty(\mathbb{D}^*)_1$, we have by using Lemma 3.5 and $\|\mu\|_\infty = \|\mu^{-1}\|_\infty$,

$$\begin{aligned}\iint_{\mathbb{D}^*} \left|D_\lambda R_\mu(\nu)\right|^2 \rho(z) d^2z \leq (1-\|\mu\|_\infty)^{-4} \iint_{\mathbb{D}^*} \left|\nu\circ w_\mu \frac{\overline{(w_\mu)_z}}{(w_\mu)_z}\right|^2 \rho(z)d^2z \\ =(1-\|\mu\|_\infty)^{-4} \iint_{\mathbb{D}^*} |\nu|^2 \frac{4|(w_{\mu^{-1}})_z|^2}{(1-|w_{\mu^{-1}}|^2)^2}(1-|\mu^{-1}|^2)d^2z \\ \leq 2(1-\|\mu\|_\infty)^{-4} \iint_{\mathbb{D}^*} |\nu|^2\rho(z)d^2z = 2(1-\|\mu|_\infty)^{-4}\|\nu\|_2^2.\end{aligned}$$

Replacing everywhere μ by μ^{-1} we get the same estimate for $D_\lambda R_{\mu^{-1}}$. □

Denote by $\mathcal{O}(\mathbb{D}^*)_1$ the subgroup of $L^\infty(\mathbb{D}^*)_1$ generated by $\mu \in \Omega^{-1,1}(\mathbb{D}^*)$, $\|\mu\|_\infty < \delta$, where δ is as in Corollary 3.6.

Lemma 3.7. *For every $\mu \in \mathcal{O}(\mathbb{D}^*)_1$ there exists $C > 0$ such that*

$$\|R_\mu(\lambda_1) - R_\mu(\lambda_2)\|_2 < C\|\lambda_1 - \lambda_2\|_2$$

for all $\lambda_1, \lambda_2 \in L^\infty(\mathbb{D}^*)_1$ satisfying $\lambda_1 - \lambda_2 \in L^2(\mathbb{D}^*, \rho(z)d^2z)$.

Proof. Suppose first that $\|\mu\|_\infty < \delta$. Set $\lambda(t) = \lambda_1 + t\nu$, where $\nu = \lambda_2 - \lambda_1$, so that $\lambda(t) \in L^\infty(\mathbb{D}^*)_1$, $0 \leq t \leq 1$. By the fundamental theorem of calculus,

$$\begin{aligned}R_\mu(\lambda_1) - R_\mu(\lambda_2) &= \int_0^1 \frac{d}{dt} R_\mu(\lambda(t))dt \\ &= \int_0^1 D_{\lambda(t)} R_\mu(\nu)dt.\end{aligned}$$

Using Corollary 3.6,

$$\begin{aligned}\|R_\mu(\lambda_1) - R_\mu(\lambda_2)\|_2^2 &= \iint_{\mathbb{D}^*} \left|\int_0^1 D_{\lambda(t)} R_\mu(\nu)(z)\right|^2 \rho(z)d^2z \\ &\leq \int_0^1 \left(\iint_{\mathbb{D}^*} \left|D_{\lambda(t)} R_\mu(\nu)(z)\right|^2 \rho(z)d^2z\right) dt \\ &\leq C^2\|\nu\|_2^2 = C^2\|\lambda_1 - \lambda_2\|_2^2.\end{aligned}$$

The same estimate also holds for R_μ^{-1}.

Since every $\mu \in \mathcal{O}(\mathbb{D}^*)_1$ can be written as $\mu_n^{\varepsilon_n} * \cdots * \mu_1^{\varepsilon_1}$, where $\mu_i \in \Omega^{-1,1}(\mathbb{D}^*)$, $\|\mu_i\|_\infty < \delta$, and $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$, we have

$$R_\mu = R_{\mu_1}^{\varepsilon_1} \circ \cdots \circ R_{\mu_n}^{\varepsilon_n},$$

and the assertion of the lemma follows. □

Remark 3.8. Applying the same argument, we get from Corollary 3.6 that for every $\mu \in \mathcal{O}(\mathbb{D}^*)_1$ there exists $C > 0$, depending only on $\|\mu\|_\infty$ such that

$$\left\| D_\lambda R_\mu(\nu) \right\|_2 \leq C\|\nu\|_2,$$

for all $\nu \in L^2(\mathbb{D}^*, \rho(z)d^2z)$ and $\lambda \in L^\infty(\mathbb{D}^*)_1$.

Lemma 3.9. *For every $\mu \in \mathcal{O}(\mathbb{D}^*)_1$ there exists $C > 0$ such that*

$$\|(\beta \circ \Phi)(\lambda * \mu) - (\beta \circ \Phi)(\mu)\|_2 \leq C\|\lambda\|_2$$

for all $\lambda \in L^2(\mathbb{D}^*,\, \rho(z)d^2z) \cap L^\infty(\mathbb{D}^*)_1$.

Proof. Set $\phi(t) = (\beta \circ \Phi)(t\lambda * \mu)$. By the fundamental theorem of calculus,

$$(\beta \circ \Phi)(\lambda * \mu) - (\beta \circ \Phi)(\mu) = \int_0^1 \frac{d\phi}{dt}(t)dt,$$

where

$$\frac{d\phi}{dt}(t) = D_{t\lambda}\left(\beta \circ \Phi \circ R_\mu\right)(\lambda) = \left(D_{t\lambda * \mu}(\beta \circ \Phi) \circ D_{t\lambda}R_\mu\right)(\lambda)$$

by the chain rule. Since $(D_0 R_\mu)^{-1} = D_\mu R_{\mu^{-1}}$, it follows from Remarks 3.4 and 3.8 that

$$\|D_{t\lambda*\mu}(\beta \circ \Phi)(\nu)\|_2 \leq 48\|D_{t\lambda*\mu}R_{(t\lambda*\mu)^{-1}}(\nu)\|_2 \leq C_1\|\nu\|_2.$$

Using Remark 3.8 again, we get

$$\left\|\frac{d\phi}{dt}(t)\right\|_2 = \|\left(D_{t\lambda*\mu}(\beta \circ \Phi) \circ D_{t\lambda}R_\mu\right)(\lambda)\|_2 \leq C_2\|\lambda\|_2, \quad 0 \leq t \leq 1.$$

Therefore,

$$\begin{aligned}\left\|(\beta \circ \Phi)(\lambda * \mu) - (\beta \circ \Phi)(\mu)\right\|_2^2 &= \iint\limits_{\mathbb{D}} \left|\int_0^1 \frac{d\phi}{dt}(t,z)dt\right|^2 \rho^{-1}(z)d^2z \\ &\leq \int_0^1 \left(\iint\limits_{\mathbb{D}} \left|\frac{d\phi}{dt}(t,z)\right|^2 \rho^{-1}(z)d^2z\right) dt \\ &\leq C_2^2\left\|\lambda\right\|_2^2,\end{aligned}$$

which concludes the proof. □

3.3. **The Hilbert manifold structure of $T(1)$.** For every $\mu \in \mathcal{O}(\mathbb{D}^*)_1$ let $V_\mu \subset U_\mu \subset T(1)$ be the image under the map $h_\mu^{-1} = \Phi \circ R_\mu \circ \Lambda$ of the open ball of radius $\sqrt{\pi/3}$ about the origin in $A_2(\mathbb{D})$, which by Lemma 3.1 is contained in the ball of radius 2 in $A_\infty(\mathbb{D})$. Here (U_μ, h_μ) is the coordinate chart U_μ of the complex-analytic atlas for $T(1)$ as a complex Banach manifold (see Section 2.1.4). Let

$$\tilde{h}_\mu = h_\mu\big|_{V_\mu} : V_\mu \to A_2(\mathbb{D}).$$

The main result of this subsection is the following.

Theorem 3.10. *For every $\mu, \nu \in \mathcal{O}(\mathbb{D}^*)_1$ the sets $\tilde{h}_\mu(V_\mu \cap V_\nu)$ and $\tilde{h}_\nu(V_\mu \cap V_\nu)$ are open in $A_2(\mathbb{D})$ and the map*

$$\tilde{h}_{\mu\nu} = \tilde{h}_\mu \circ \tilde{h}_\nu^{-1} : \tilde{h}_\nu(V_\mu \cap V_\nu) \longrightarrow \tilde{h}_\mu(V_\mu \cap V_\nu) \subset A_2(\mathbb{D})$$

is a biholomorphic function in the Hilbert space $A_2(\mathbb{D})$.

Proof. First we prove that the sets $\tilde{h}_\mu(V_\mu \cap V_\nu)$ and $\tilde{h}_\nu(V_\mu \cap V_\nu)$ are open in $A_2(\mathbb{D})$. Since $V_\mu \cap V_\nu \neq \emptyset$ (otherwise there is nothing to prove), there exist $\phi_1 \in \tilde{h}_\mu(V_\mu \cap V_\nu)$ and $\phi_2 \in \tilde{h}_\nu(V_\mu \cap V_\nu)$, $\|\phi_1\|_2, \|\phi_2\|_2 < \sqrt{\pi/3}$, such that $\tilde{h}_\mu^{-1}(\phi_1) = \tilde{h}_\nu^{-1}(\phi_2)$, i.e.,

$$(\Phi \circ R_\mu \circ \Lambda)(\phi_1) = (\Phi \circ R_\nu \circ \Lambda)(\phi_2).$$

Setting $\lambda_1 = \Lambda(\phi_1), \lambda_2 = \Lambda(\phi_2)$ and $\kappa = \nu * \mu^{-1}$, we get

$$\Phi(\lambda_1) = \Phi(\lambda_2 * \kappa).$$

The sets $h_\mu(U_\mu \cap U_\nu)$ and $h_\nu(U_\mu \cap U_\nu)$ are open in $A_\infty(\mathbb{D})$, so that there exists $\delta_1 > 0$ such that $h_\mu(U_\mu \cap U_\nu)$ contains a ball of radius δ_1 about ϕ_1 in $A_\infty(\mathbb{D})$. The mapping $h_{\mu\nu} : h_\nu(U_\mu \cap U_\nu) \to h_\mu(U_\mu \cap U_\nu) \subset A_\infty(\mathbb{D})$ is a continuous function in the Banach space $A_\infty(\mathbb{D})$, so that there exists $\delta_2 > 0$ such that the inverse image by $h_{\mu\nu}$ of the ball of radius δ_1 about ϕ_1 in $A_\infty(\mathbb{D})$ contains the ball of radius δ_2 about ϕ_2 in $A_\infty(\mathbb{D})$. According to Lemma 3.1, the latter ball contains any ball of radius $\delta_3 < \sqrt{\pi/12}\,\delta_2$ about ϕ_2 in $A_2(\mathbb{D})$. Now for every $\varphi_2 \in A_2(\mathbb{D})$ satisfying $\|\varphi_2 - \phi_2\|_2 < \delta_3$ set

$$\varphi_1 = h_{\mu\nu}(\varphi_2) = (\beta \circ \Phi \circ R_\kappa \circ \Lambda)(\varphi_2).$$

We claim that $\delta_3 > 0$ can be chosen such that $\varphi_1 \in A_2(\mathbb{D})$ and $\|\varphi_1\|_2 < \sqrt{\pi/3}$, which implies that $\tilde{h}_\nu(V_\mu \cap V_\nu)$ contains the ball of radius δ_3 about ϕ_2 in $A_2(\mathbb{D})$. Indeed, set $\lambda = \Lambda(\varphi_2)$, so that $\varphi_1 = (\beta \circ \Phi)(\lambda * \kappa)$. Since $\lambda - \lambda_2 \in L^2(\mathbb{D}^*, \rho(z)d^2z)$, we have by Lemmas 3.9 and 3.7,

$$\begin{aligned}\|\varphi_1 - \phi_1\|_2 &= \|(\beta \circ \Phi)(\lambda * \kappa) - (\beta \circ \Phi)(\lambda_2 * \kappa)\|_2 \\ &\leq C\|\lambda * \lambda_2^{-1}\|_2 \leq C^2\|\lambda - \lambda_2\|_2 \\ &= 2C^2\|\varphi_2 - \phi_2\|_2 < 2C^2\delta_3,\end{aligned}$$

where the constant $C > 0$ (chosen to be the same for both Lemmas 3.7 and 3.9) depends only on λ_2 and κ. Choosing δ_3 small enough we have $\|\varphi_1\|_2 < \sqrt{\pi/3}$.

The same argument applied to the map $\tilde{h}_{\nu\mu} = \tilde{h}_{\mu\nu}^{-1}$ proves that $\tilde{h}_\mu(V_\mu \cap V_\nu)$ is open in $A_2(\mathbb{D})$.

It remains to prove that the map $\tilde{h}_{\mu\nu}$ is a holomorphic function in the Hilbert space $A_2(\mathbb{D})$. It is bounded, so according to [Bou67] it is sufficient to prove that for every $\lambda \in \tilde{h}_\nu(V_\mu \cap V_\nu)$ and every $\eta \in A_2(\mathbb{D})$ the mapping $\mathbb{C} \ni t \mapsto \phi(t) = \tilde{h}_{\mu\nu}(\lambda + t\eta) \in A_2(\mathbb{D})$ is a holomorphic function in some neighborhood of 0 in $\mathbb{C}$. For this purpose we use the standard argument based on the fact that the map $h_{\mu\nu}$ is already a holomorphic function in the Banach space $A_\infty(\mathbb{D})$ and the mapping $\mathbb{C} \ni t \mapsto \phi(t) \in A_\infty(\mathbb{D})$ is a holomorphic function in some neighborhood of 0 in $\mathbb{C}$. Thus there exists $\delta > 0$ such that for every $|t_0| < \delta$,

$$\left\| \phi(t) - \phi(t_0) - (t-t_0)\frac{d\phi}{dt}(t_0) \right\|_\infty = o(|t-t_0|) \quad \text{as} \quad t \to t_0.$$

Moreover, δ can be chosen such that $\lambda + t\eta \in \tilde{h}_\nu(V_\mu \cap V_\nu)$ for $|t| < \delta$. Then for every $z \in \mathbb{D}$ the complex-valued function $\phi(t)(z)$ is holomorphic on $|t| < \delta$ and

$$\begin{aligned} &\phi(t,z) - \phi(t_0,z) - (t-t_0)\frac{d\phi}{dt}(t_0,z) \\ = &\frac{1}{2\pi i}\oint_{|w-t_0|=\delta_1} \phi(w,z)\left(\frac{1}{w-t} - \frac{1}{w-t_0} - \frac{t-t_0}{(w-t_0)^2}\right) dw \\ &= \frac{(t-t_0)^2}{2\pi i}\oint_{|w-t_0|=\delta_1} \frac{\phi(w,z)}{(w-t_0)^2(w-t)} dw, \end{aligned}$$

where $\delta_1 > 0$ is such that the disk of radius δ_1 about t_0 is inside the disk of radius δ about the origin, and t satisfies $|t - t_0| < \delta_1$. Since $\phi(t) \in \tilde{h}_\mu(V_\mu \cap V_\nu)$, $\|\phi(t)\|_2^2 < \pi/3$ for all $|t| < \delta$, and we have

$$\begin{aligned} &\left\| \frac{\phi(t)-\phi(t_0)}{t-t_0} - \frac{d\phi}{dt}(t_0) \right\|_2^2 \\ \leq &\frac{|t-t_0|^2}{4\pi^2}\oint_{|w|=\delta_1} \frac{|dw|}{|w|^4|w-(t-t_0)|^2}\oint_{|w|=\delta_1} \left\|\phi(w+t_0)\right\|_2^2 |dw| \\ = &O(|t-t_0|^2) \quad \text{as} \quad t \to t_0. \end{aligned}$$

□

According to [Bou67], Theorem 3.10 justifies the following definition.

Definition 3.11. The covering

$$T(1) = \bigcup_{\mu \in \mathcal{O}(\mathbb{D}^*)_1} V_\mu$$

with the coordinate maps $\tilde{h}_\mu : V_\mu \longrightarrow A_2(\mathbb{D})$ and the transition maps

$$\tilde{h}_{\mu\nu} = \tilde{h}_\mu \circ \tilde{h}_\nu^{-1} : \tilde{h}_\nu(V_\mu \cap V_\nu) \longrightarrow \tilde{h}_\mu(V_\mu \cap V_\nu)$$

is a complex-analytic atlas which endows $T(1)$ with the structure of a complex Hilbert manifold modeled on the Hilbert space $A_2(\mathbb{D})$.

Corollary 3.12. *The right translations are biholomorphic mappings on the Hilbert manifold $T(1)$.*

Proof. Representing every point in $T(1)$ by $\mu \in \mathcal{O}(\mathbb{D}^*)_1$ we have $R_{[\mu]}(V_\lambda) = V_{\lambda * \mu}$, so that $\tilde{h}_{\lambda*\mu} \circ R_{[\mu]} \circ \tilde{h}_\lambda^{-1}$ is the identity mapping on $\tilde{h}_\lambda(V_\lambda) \subset A_2(\mathbb{D})$. □

We will continue to use the name Bers coordinates for the complex coordinates $(V_\mu, \tilde{h}_\mu)$ on the Hilbert manifold $T(1)$. As in Section 2.1.4, the vector field $\frac{\partial}{\partial \varepsilon_\nu}$ corresponding to $\nu \in H^{-1,1}(\mathbb{D}^*)$ at a point $[\mu] \in V_0$ in terms of the Bers coordinates on V_μ has the same form (2.5), i.e.,

$$\left.\frac{\partial}{\partial \varepsilon_\nu}\right|_{[\mu]} = P\left(\left(\frac{\nu}{1-|\mu|^2}\,\frac{(w_\mu)_z}{(\overline{w}_\mu)_{\bar{z}}}\right) \circ w_\mu^{-1}\right),$$

where $P : L^2(\mathbb{D}^*, \rho(z)d^2z) \to H^{-1,1}(\mathbb{D}^*)$ is the orthogonal projector given by (2.7).

3.4. **Integral manifolds of the distribution $\mathfrak{D}_T$.** Finally, we introduce a Hilbert manifold structure on the Banach space $A_\infty(\mathbb{D})$ by defining the coordinate chart at every $\phi \in A_\infty(\mathbb{D})$ to be $\phi + A_2(\mathbb{D})$. By Lemma 3.1 the Hilbert manifold topology on $A_\infty(\mathbb{D})$ is stronger than the Banach space topology. The Hilbert manifold $A_\infty(\mathbb{D})$ is not connected. Rather $A_\infty(\mathbb{D})$ is the union of uncountably many components $\phi + A_2(\mathbb{D})$ with $\phi \in A_\infty(\mathbb{D})/A_2(\mathbb{D})$, which are integral manifolds of the distribution $\mathfrak{D}_A$.

Theorem 3.13. *The Bers embedding $\beta : T(1) \to \beta(T(1)) \subset A_\infty(\mathbb{D})$ is a biholomorphic mapping of Hilbert manifolds.*

Proof. To prove that the Bers embedding is holomorphic it is sufficient to show that for every $\mu \in \mathcal{O}(\mathbb{D}^*)_1$ that image of the ball of radius $\sqrt{\pi/3}$ about 0 in $A_2(\mathbb{D})$ by the mapping $\beta \circ \tilde{h}_\mu^{-1} = \beta \circ \Phi \circ R_\mu \circ \Lambda$ is inside a translate by $(\beta \circ \Phi)(\mu)$ of some ball about 0 in $A_2(\mathbb{D})$. This immediately follows from Lemma 3.9,

$$\begin{aligned}\left\|\left(\beta \circ \tilde{h}_\mu^{-1}\right)(\varphi) - (\beta \circ \Phi)(\mu)\right\|_2 &= \|(\beta \circ \Phi)(\lambda * \mu) - (\beta \circ \Phi)(\mu)\|_2 \\ &< C\|\lambda\|_2,\end{aligned}$$

where $\lambda = \Lambda(\varphi) \in \mathcal{O}(\mathbb{D}^*)_1$ and the constant $C > 0$ depends only on $\|\mu\|_\infty$. Since the Bers embedding is holomorphic mapping of Banach manifolds, the standard argument used in the proof of Theorem 3.10 works for this case, so that the mapping $\beta \circ \tilde{h}_\mu^{-1} - (\beta \circ \Phi)(\mu)$ is a holomorphic function in the Hilbert space $A_2(\mathbb{D})$.

Finally, the image $\beta(T(1))$ is open in the Hilbert manifold $A_\infty(\mathbb{D})$ since it is open in a weaker Banach manifold topology. Using Theorem 3.3 and the inverse function theorem for Hilbert manifolds [Lan95] we see that the Bers embedding is biholomorphic. □

Theorem 3.13 allows to conclude that the distribution $\mathfrak{D}_T$ on $T(1)$ is equivalent to the restriction of the distribution $\mathfrak{D}_A$ on $\beta(T(1)) \subset A_\infty(\mathbb{D})$ and, therefore, is integrable. Its integral manifolds are inverse images by the Bers embedding β of the integral manifolds of the distribution $\mathfrak{D}_A$ on $\beta(T(1))$, i.e., of the components $(\phi + A_2(\mathbb{D})) \cap \beta(T(1))$. For every $[\mu] \in T(1)$ denote by $T_{[\mu]}(1)$ the component of the Hilbert manifold $T(1)$ containing $[\mu]$. It follows from Theorems 3.3 and 3.13 that the Hilbert manifold $T_{[\mu]}(1)$ is the integral manifold of the distribution $\mathfrak{D}_T$ passing through $[\mu] \in T(1)$. The right translations act transitively on the set of components, $R_{[\nu]}(T_{[\mu]}(1)) = T_{[\mu*\nu]}(1)$ for all $[\mu], [\nu] \in T(1)$.

4. Velling-Kirillov and Weil-Petersson metrics

4.1. **Velling-Kirillov metric on the universal Teichmüller curve.** The Velling-Kirillov metric is a right-invariant Hermitian metric on $\mathcal{T}(1)$, defined at the origin of $\mathcal{T}(1)$ by

$$\|\mathrm{v}\|_{VK}^2 = \sum_{n=1}^{\infty} n|c_n|^2, \tag{4.1}$$

where

$$\mathrm{v} = \frac{\mathrm{u} - iJ\,\mathrm{u}}{2} \quad \text{and} \quad \mathrm{u} = \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n e^{in\theta}\frac{d}{d\theta} \in T_0^{\mathbb{R}} S^1\backslash\mathrm{Homeo}_{qs}(S^1).$$

The convergence of the series is guaranteed by the property **TS4** (with $s = 1/2$) in Section 2.2.3. The Velling-Kirillov metric is a smooth right-invariant Kähler metric on the complex Banach manifold $\mathcal{T}(1)$. Its symplectic form ω_{VK} at the origin of $\mathcal{T}(1)$ is given by

$$\omega_{VK}(\mathrm{v}, \bar{\mathrm{v}}) = \frac{i}{2}\|\mathrm{v}\|_{VK}^2,$$

In the next section we prove that the Velling-Kirillov metric is real-analytic on $\mathcal{T}(1)$ by presenting its real-analytic Kähler potential.

Remark 4.1. For the homogeneous space $S^1\backslash\mathrm{Diff}_+(S^1)$ the metric was introduced in this form by A.A. Kirillov [Kir87] and it has been studied by A.A. Kirillov and D. Yuriev. In particular, in [KY87] it was shown to be Kähler . In [Vel], J. Velling has introduced a Hermitian metric for the space $\mathcal{T}(1)$ using arguments from the geometric theory of functions. In [Teo02], Kirillov's definition was extended to $\mathcal{T}(1)$ and it was shown that the resulting metric coincides with the metric introduced by Velling. The Velling-Kirillov metric is the unique right-invariant Kähler metric on the universal Teichmüller curve $\mathcal{T}(1)$ [Kir87, Teo02].

4.2. **Weil-Petersson metric on the universal Teichmüller space.** In this section we consider $T(1)$ as a Hilbert manifold. The Weil-Petersson metric on $T(1)$ is a Hermitian metric defined by the Hilbert space inner product on tangent spaces, which are identified with the Hilbert space $H^{-1,1}(\mathbb{D}^*)$ by right translations (see Section 3.3). Thus the Weil-Petersson metric is a right-invariant metric on $T(1)$ defined at the origin of $T(1)$ by

$$\langle\mu,\nu\rangle_{WP} = \iint\limits_{\mathbb{D}^*} \mu\bar{\nu}\rho(z)d^2z, \quad \mu,\nu \in H^{-1,1}(\mathbb{D}^*) = T_0T(1). \tag{4.2}$$

To every $\mu \in H^{-1,1}(\mathbb{D}^*)$ there corresponds a vector field $\frac{\partial}{\partial \varepsilon_\mu}$ over V_0, given by (2.5)-(2.7). We set for every $\kappa \in V_0$,

$$g_{\mu\bar{\nu}}(\kappa) = \left\langle \frac{\partial}{\partial\varepsilon_\mu}\Big|_\kappa, \frac{\partial}{\partial\varepsilon_\nu}\Big|_\kappa \right\rangle_{WP} = \iint\limits_{\mathbb{D}^*} P(R(\mu,\kappa))\overline{P(R(\nu,\kappa))}\rho(z)d^2z. \tag{4.3}$$

This formula explicitly defines the Weil-Petersson metric on the coordinate chart V_0. The Weil-Petersson metric extends to other charts V_μ by right translations.

The following statement is an easy consequence of Lemma 3.5.

Lemma 4.2. *The Weil-Petersson metric is continuous on $T(1)$.*

Proof. As it follows from Corollary 3.12, it is sufficient to prove that for every $\mu \in H^{-1,1}(\mathbb{D}^*)$ the function $g_{\mu\bar{\mu}}$ is continuous on V_0 at 0. Since the embedding $V_0 \hookrightarrow U_0$ is continuous by Lemma 3.1, it is sufficient to prove that the function $g_{\mu\bar{\mu}}$ is defined on the neighborhood of 0 in U_0 and is continuous at 0.

Since the projector P is norm-decreasing,

$$\begin{aligned} g_{\mu\bar{\mu}}(\kappa) &= \iint\limits_{\mathbb{D}^*} P(R(\mu,\kappa))\overline{P(R(\mu,\kappa))}\rho(z)d^2z \\ &\leq \iint\limits_{\mathbb{D}^*} R(\mu,\kappa)\overline{R(\mu,\kappa)}\rho(z)d^2z \\ &= 4\iint\limits_{\mathbb{D}^*} \frac{|\mu|^2}{1-|\kappa|^2}\frac{|(w_\kappa)_z|^2}{(1-|w_\kappa|^2)^2}d^2z. \end{aligned}$$

According to Lemma 3.5, for every $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for all $\kappa \in U_0$ satisfying $\|\kappa\|_\infty < \delta$ we have

$$\begin{aligned} \Big|g_{\mu\bar{\mu}}(\kappa) - g_{\mu\bar{\mu}}(0)\Big| &\leq 4\iint\limits_{\mathbb{D}^*} \frac{|\mu|^2}{1-|\kappa|^2}\left|\frac{|(w_\kappa)_z|^2}{(1-|w_\kappa|^2)^2} - \frac{1}{(1-|z|^2)^2}\right| \\ &\qquad + \frac{|\mu|^2}{(1-|z|^2)^2}\left(\frac{1}{1-|\kappa|^2} - 1\right)d^2z \\ &\leq \frac{\varepsilon+\delta^2}{1-\delta^2}\iint\limits_{\mathbb{D}} |\mu|^2\rho d^2z. \end{aligned}$$

Thus, for δ small enough $\Big|g_{\mu\bar{\mu}}(\kappa) - g_{\mu\bar{\mu}}(0)\Big| \leq 2\varepsilon$. □

Remark 4.3. Using the basic properties of the q.c. mappings, it can be shown that the Weil-Petersson metric is real-analytic on $T(1)$. In fact, it is sufficient to prove that for every $\mu,\nu \in H^{-1,1}(\mathbb{D}^*)$ the mapping $V_0 \ni \kappa \mapsto g_{\mu\bar{\nu}}(\kappa) \in \mathbb{C}$ is real-analytic on V_0. Since this result will not be used later, we omit the proof. Explicit curvature computations in Section 7 will show that the Weil-Petersson metric on $T(1)$ is twice differentiable.

We will prove in Section 7 that the Weil-Petersson metric is Kähler . Its symplectic form ω_{WP} is a right-invariant $(1,1)$ form on the Hilbert manifold $T(1)$. At the origin of $T(1)$,

$$\omega_{WP}(\mu,\bar{\nu})=\tfrac{i}{2}\langle\mu,\nu\rangle_{WP},\quad \mu,\nu\in H^{-1,1}(\mathbb{D}^*).$$

Remark 4.4. The Weil-Petersson metric on the distribution $\mathfrak{D}_T$ (without defining the Hilbert manifold structure) was introduced by S. Nag and A. Verjovsky [NV90] as a direct generalization of the Weil-Petersson metric on the finite-dimensional Teichmüller spaces. It was proved in [NV90] that the embedding $\text{Möb}(S^1)\backslash\text{Diff}_+(S^1)\hookrightarrow T(1)$ is holomorphic and the pull-back of the Weil-Petersson metric on the distribution $\mathfrak{D}_T$ coincides, up to a constant, with the right invariant Kähler metric introduced by Kirillov [Kir87] by the orbit method. At the tangent space of the origin the latter metric is defined by (cf. (4.1))

$$\|\mathrm{v}\|^2=\sum_{n=2}^{\infty}(n^3-n)|c_n|^2,$$

where

$$\mathrm{v}=\frac{\mathrm{u}-iJ\,\mathrm{u}}{2}\quad\text{and}\quad \mathrm{u}=\sum_{n\in\mathbb{Z}\backslash\{0,\pm1\}}c_ne^{in\theta}\frac{d}{d\theta}\in T_0^{\mathbb{R}}\,\text{Möb}(S^1)\backslash\text{Diff}_+(S^1).$$

It follows from the results in Section 3.3 that the closure of $\text{Möb}(S^1)\backslash\text{Diff}_+(S^1)$ in the Hilbert manifold topology on $T(1)$ is the Hilbert submanifold $T_{[0]}(1)$.

5. Characteristic forms of the universal Teichmüller curve

Let $V=T_v\mathcal{T}(1)$ be the vertical tangent bundle of the fibration

$$\pi:\mathcal{T}(1)\to T(1).$$

It is a holomorphic line bundle over the complex Banach manifold $\mathcal{T}(1)$, the fiber over a point $([\mu],z)\in\mathcal{T}(1)$ is a holomorphic tangent bundle to the quasi-disc $w^\mu(\mathbb{D}^*)$. The hyperbolic metric on $w^\mu(\mathbb{D}^*)$ defines a Hermitian metric on V, and we denote by $c_1(V)$ the first Chern form of V corresponding to this metric.

The Hilbert manifold structure on $T(1)$ naturally induces a Hilbert manifold structure on $\mathcal{T}(1)$, such that the projection $\pi:\mathcal{T}(1)\to T(1)$ is a holomorphic mapping of Hilbert manifolds, and the vertical tangent bundle is a holomorphic line bundle over the Hilbert manifold $T(1)$. We will continue to denote corresponding line bundle by V, and by $c_1(V)$ — the first Chern form corresponding to the hyperbolic metric on the fibers, specifying explicitly which topology we are using. Since the Hilbert manifold topology is stronger than the Banach manifold topology, the form $c_1(V)$ for the Banach manifold structure on $\mathcal{T}(1)$ naturally restricts onto $\mathcal{T}(1)$ considered as the Hilbert manifold.

Similar to Wolpert's work [Wol86] on finite dimensional Teichmüller spaces, we define the analogs of Mumford-Morita-Miller characteristic forms as the following (n,n)-forms on the Hilbert manifold $T(1)$,

$$\kappa_n = (-1)^{n+1}\pi_*(c_1(V)^{n+1}), \tag{5.1}$$

where $\pi_* : \Omega^*(\mathcal{T}(1)) \to \Omega^{*-2}(T(1))$ is the operation of "integration over the fibers" of the fibration $\pi : \mathcal{T}(1) \to T(1)$. As we will see in Section 5.3, it is the passage from the Banach manifold structure to the Hilbert manifold structure that makes the operation π_* well-defined (i.e., the integrals over non-compact fibers become convergent).

5.1. **The form $c_1(V)$ as Velling-Kirillov symplectic form.** In this section we work with the Banach manifold structure on $\mathcal{T}(1)$. Let z be the complex coordinate on $\hat{\mathbb{C}}\setminus\{0\}$. The assignment $\mathcal{T}(1) \ni ([\mu], z) \mapsto -z^2\partial_z$ defines a holomorphic section of the line bundle V over $\mathcal{T}(1)$ [6]. The hyperbolic metric on $w^\mu(\mathbb{D}^*)$ is a pull-back of the hyperbolic metric on $\mathbb{D}^*$ by the conformal map g_μ, so that

$$\left\|z^2\partial_z\right\|^2_{([\mu],z)} = \frac{4|z^2(g_\mu^{-1})'(z)|^2}{(|g_\mu^{-1}(z)|^2-1)^2}.$$

The first Chern form of the line bundle V is

$$c_1(V) = \frac{i}{2\pi}\Theta = \frac{i}{2\pi}\bar\partial\partial\log\left\|z^2\partial_z\right\|^2,$$

where ∂ and $\bar\partial$ are, respectively, the $(1,0)$ and $(0,1)$ components of the de Rham differential on $\mathcal{T}(1)$.

Let

$$K = \log\left\|z^2\partial_z\right\|_{([\mu],z)} - \log 2.$$

Lemma 5.1. *The function $K : \mathcal{T}(1) \to \mathbb{R}$ is real-analytic. Under the correspondence $\mathcal{T}(1) \ni ([\mu], z) \mapsto \gamma \in S^1\backslash Homeo_{qs}(S^1)$, where $\gamma = (g^{-1}\circ f)\big|_{S^1}$,*

$$K(\gamma) = \log|g'(\infty)|.$$

Proof. Using formulas $g = \lambda_w \circ g_\mu \circ \sigma_w^{-1}$ and $w = g_\mu^{-1}(z)$ from Section 2.2.1, it is straightforward to compute

$$g'(\infty) = \frac{z^2(g_\mu^{-1})'(z)(1-\bar w)}{(|w|^2-1)(1-w)}.$$

Now it easily follows from the general properties of q.c. mappings that for $z \in \mathbb{D}^*$ the functional $T(1) \ni [\mu] \mapsto g_\mu^{-1}(z) \in \mathbb{C}$ is real-analytic so that $|g'(\infty)|$ is real-analytic function on $\mathcal{T}(1)$. ☐

Remark 5.2. The quantity $|g'(\infty)|$ is the capacity of the quasi-circle $g(S^1)$ corresponding to $\gamma \in \mathrm{Homeo}_{qs}(S^1)$.

[6] Under the conformal map $z \mapsto \frac{1}{z}$ the vector field $-z^2\partial_z \mapsto \partial_z$.

Theorem 5.3. *The first Chern form of the vertical tangent bundle to the universal Teichmüller curve $\mathcal{T}(1)$ is proportional to the symplectic form of the Velling-Kirillov metric,*

$$c_1(V) = -\tfrac{2}{\pi}\,\omega_{VK}.$$

Equivalently, the function K is a Kähler potential for the Velling-Kirillov metric on $\mathcal{T}(1)$.

Proof. It is based on the following lemmas.

Lemma 5.4. *The $(1,1)$-form $c_1(V)$ on $\mathcal{T}(1)$ is right-invariant.*

Proof. We need to proof that for every $\gamma_0 \in S^1\backslash\mathrm{Homeo}(S^1) \cong \mathcal{T}(1)$ the difference $R^*_{\gamma_0}K - K$, where $R_{\gamma_0} : \mathcal{T}(1) \to \mathcal{T}(1)$ is a right translation and $(R^*_{\gamma_0}K)(\gamma) = K(\gamma\circ\gamma_0)$, is a harmonic function on $\mathcal{T}(1)$.

For every $\gamma = g^{-1}\circ f \in \mathcal{T}(1)$ let $\tilde{\gamma} = \gamma\circ\gamma_0 = \tilde{g}^{-1}\circ\tilde{f}$. Since $\tilde{g} = \tilde{f}\circ\gamma_0^{-1}\circ f^{-1}\circ g$, we have

$$\left(R^*_{\gamma_0}K - K\right)(\gamma) = \log|\tilde{g}'(\infty)| - \log|g'(\infty)| = \log|(\tilde{f}\circ\gamma_0^{-1}\circ f^{-1})'(\infty)|.$$

In [Ber73], Bers has proved that the function

$$(\gamma, z) \mapsto \left(\tilde{f}\circ\gamma_0^{-1}\circ f^{-1}\right)(z) = h(\gamma, z)$$

depends holomorphically on γ and z, which implies that $\left(\tilde{f}\circ\gamma_0^{-1}\circ f^{-1}\right)'(\infty)$ depends holomorphically on γ and our assertion follows. □

Lemma 5.5. *Let $\gamma = g^{-1}\circ f \in \mathcal{T}(1)$, where $f|_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} a_n z^{n+1}$, $a_0 = 1$, and $g|_{\mathbb{D}^*}(z) = \sum_{n=0}^{\infty} b_n z^{1-n}$. Then*

$$|b_0|^2 = \sum_{n=0}^{\infty}(n+1)|a_n|^2 + \sum_{n=1}^{\infty}(n-1)|b_n|^2. \tag{5.2}$$

Proof. Evaluate the Euclidean area of the domain $\Omega = f(\mathbb{D}) = g(\mathbb{D})$ in two different ways. First,

$$A_E(\Omega) = \lim_{r\to 1^-}\iint\limits_{f(\mathbb{D}_r)} d^2z = \lim_{r\to 1^-}\iint\limits_{\mathbb{D}_r} |f'|^2 d^2z = \pi\sum_{n=0}^{\infty}(n+1)|a_n|^2,$$

where $\mathbb{D}_r = \{z\in\mathbb{C} : |z| < r\}$. On the other hand, the classical area theorem gives

$$A_E(\Omega) = \pi\sum_{n=0}^{\infty}(1-n)|b_n|^2,$$

and we obtain (5.2). □

Now we complete the proof of the theorem. Let

$$\mathrm{u} = \sum_{n\in\mathbb{Z}\backslash\{0\}} c_n e^{in\theta}\frac{d}{d\theta} \in T_0^{\mathbb{R}}\mathcal{T}(1),$$

and let $\gamma_t = g_t^{-1} \circ f^t$, $\gamma_0 = \mathrm{id}$, be corresponding smooth curve in $\mathcal{T}(1)$. Using notations from the previous lemma, differentiate the relation (5.2) with respect to t and set $t = 0$. Denoting

$$\dot{a}_n(\mathrm{u}) = \dot{a}_n = \frac{da_n}{dt}(0) \quad \text{and} \quad \dot{b}_n(\mathrm{u}) = \dot{b}_n = \frac{db_n}{dt}(0)$$

and using $b_0(0) = 1$, $a_n(0) = b_n(0) = 0$ for $n \geq 1$, we get

$$\dot{b}_0 + \bar{\dot{b}}_0 = 0. \tag{5.3}$$

Differentiating (5.2) twice with respect to t and setting $t = 0$ we get

$$\begin{aligned}\frac{d^2 b_0}{dt^2}(0) + \frac{d^2 \bar{b}_0}{dt^2}(0) + 2\left|\frac{db_0}{dt}(0)\right|^2 &= 2\sum_{n=1}^{\infty}(n+1)|\dot{a}_n|^2 + 2\sum_{n=1}^{\infty}(n-1)|\dot{b}_n|^2 \\ &= 4\sum_{n=1}^{\infty} n|\dot{a}_n|^2,\end{aligned}$$

where we have also used the property **TS1** in Section 2.2.3. Since $g_t'(\infty) = b_0(t)$, using (5.3) we get

$$\frac{d^2}{dt^2}\log|g_t'(\infty)|\big|_{t=0} = 2\sum_{n=1}^{\infty} n|\dot{a}_n|^2.$$

Let $\mathrm{v} = \frac{1}{2}(\mathrm{u} - iJ\,\mathrm{u})$ be the holomorphic tangent vector to $\mathcal{T}(1)$. Since $\dot{a}_n(J\,\mathrm{u}) = i\dot{a}_n(\mathrm{u})$, using (2.15) we finally get

$$(\partial\bar{\partial}K)(\mathrm{v}, \bar{\mathrm{v}}) = \frac{1}{2}\sum_{n=1}^{\infty} n\left(|\dot{a}_n(\mathrm{u})|^2 + |\dot{a}_n(J\,\mathrm{u})|^2\right) = \sum_{n=1}^{\infty} n|\dot{a}_n(\mathrm{u})|^2. \tag{5.4}$$

This proves that $\Theta = 4i\omega_{VK}$ at the origin of $\mathcal{T}(1)$. Since both these $(1,1)$-forms on $\mathcal{T}(1)$ are right-invariant, the assertion follows. □

Remark 5.6. In [KY87], A.A. Kirillov and D. Yuriev have stated that the function K, restricted to the space $S^1\backslash\mathrm{Diff}_+(S^1)$, is a Kähler potential of the Velling-Kirillov metric. Theorem 5.3 extends this result to $\mathcal{T}(1) \simeq S^1\backslash\mathrm{Homeo}_{qs}(S^1)$ and gives its geometric interpretation.

5.2. **The Chern form $c_1(V)$ and the resolvent kernel.** Let $\mu \in \Omega^{-1,1}(\mathbb{D}^*)$ be a horizontal holomorphic tangent vector to $\mathcal{T}(1)$ at the origin. According to the property **TV1** in Section 2.2.2, the vector field τ_μ — the horizontal lift of the vector field $\frac{\partial}{\partial\varepsilon_\mu}$ on $U_0 \subset T(1)$ to the point $(0, z) \in \pi^{-1}(0)$, is identified with $(\sigma_z^{-1})^*\mu \in \Omega^{-1,1}(\mathbb{D}^*)$.

Proposition 5.7. (i) *On the fiber $\pi^{-1}(0) \subset \mathcal{T}(1)$ the Velling-Kirillov metric is given by*

$$\begin{aligned} \langle \partial_z, \partial_z \rangle_{VK}(0,z) &= \frac{1}{(1-|z|^2)^2}, \\ \langle \partial_z, \tau_\mu \rangle_{VK}(0,z) &= 0, \\ \langle \tau_\mu, \tau_\mu \rangle_{VK}(0,z) &= \frac{1}{2} \iint\limits_{\mathbb{D}^*} G(z,u)|\mu(u)|^2 \rho(u) d^2u. \end{aligned}$$

(ii) *On the fiber $\pi^{-1}(0) \subset \mathcal{T}(1)$ the $(1,1)$-form Θ is given by*

$$\begin{aligned} \Theta(\partial_z, \partial_{\bar z})(0,z) &= -\frac{2}{(1-|z|^2)^2}, \\ \Theta(\partial_z, \tau_{\bar\mu})(0,z) &= 0, \\ \Theta(\tau_\mu, \tau_{\bar\mu})(0,z) &= -\iint\limits_{\mathbb{D}^*} G(z,u)|\mu(u)|^2 \rho(u) d^2u. \end{aligned}$$

(iii) *The vertical holomorphic tangent bundle $V \to \mathcal{T}(1)$ of the fibration $\pi : \mathcal{T}(1) \to T(1)$ is a negative line bundle.*

Proof. It follows from the property **TV2** in Section 2.2.2 that the vector field ∂_z at $(0,z) \in \pi^{-1}(0)$ corresponds to the tangent vector

$$\mathrm{u} = \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n e^{in\theta} \frac{d}{d\theta} \in T_0^{\mathbb{R}} S^1 \backslash \mathrm{Homeo}_{qs}(S^1)$$

with $c_1 = \frac{1-\bar z}{(1-z)(1-|z|^2)}$ and $c_n = 0$ for $n \geq 2$. This proves the first formula in part (i). The second formula follows from the fact that, according to the property **TS1** in Section 2.2.3, the tangent vector $\mathrm{u} \in T_0^{\mathbb{R}} S^1 \backslash \mathrm{Homeo}_{qs}(S^1)$, which corresponds to the horizontal lift τ_μ of the vector field $\frac{\partial}{\partial \varepsilon_\mu}$ to $(0,z) \in \pi^{-1}(0)$, has $c_1 = 0$. The last formula follows from the following lemma.

Lemma 5.8. *Let $\mu \in \Omega^{-1,1}(\mathbb{D}^*)$ and*

$$\phi(z) = D_0\beta(\mu)(z) = \sum_{n=2}^{\infty} (n^3 - n) a_n z^{n-2} \in A_\infty(\mathbb{D}).$$

Then

$$\iint\limits_{\mathbb{D}^*} G(z,u)|\mu(u)|^2 \rho(u) d^2u = 2 \sum_{n=2}^{\infty} n |a_n^z|^2,$$

where

$$(\sigma_z^{-1})^*(\phi)(u) = \sum_{n=2}^{\infty} (n^3 - n) a_n^z u^{n-2}$$

is the power series expansion of $(\sigma_z^{-1})^*(\phi) = \phi \circ \sigma_z^{-1} \left((\sigma_z^{-1})'\right)^2 \in A_\infty(\mathbb{D})$.

Proof. Since $G(z,u)$ is a point-pair invariant, it is sufficient to prove the formula only for $z=\infty$. In this case, using the relation $G(\infty,u)=G(0,1/\bar{u})$ between the resolvent kernels on $\mathbb{D}^*$ and $\mathbb{D}$ and the formula $\mu=\Lambda(\phi)$ we get

$$\iint_{\mathbb{D}^*} G(\infty,u)|\mu(u)|^2\rho(u)d^2u = \iint_{\mathbb{D}} G(0,u)(1-|u|^2)^2|\phi(u)|^2d^2u.$$

It follows from the explicit formula (2.18) that

$$(1-|u|^2)^2G(0,u)=h(r)=\left(\frac{1}{2\pi}\frac{1+r^2}{1-r^2}\log\frac{1}{r^2}-\frac{1}{\pi}\right)(1-r^2)^2,\quad r=|u|.$$

Since h is an integrable function on $[0,1)$, we have

$$\iint_{\mathbb{D}} G(0,u)(1-|u|^2)^2|\phi(u)|^2d^2u=2\pi\sum_{n=2}^{\infty}(n^3-n)^2|a_n|^2\int_0^1 h(r)r^{2n-4}rdr.$$

A straightforward computation gives

$$\int_0^1 h(r)r^{2n-4}rdr=\frac{1}{\pi}\frac{1}{(n^3-n)(n^2-1)},$$

which proves the lemma. □

Using this Lemma, the property **TV1** in Section 2.2.2 and the property **TS2** in Section 2.2.3, we get the third formula in part (i). Now part (ii) follows from Theorem 5.3, and part (iii) follows from part (ii) and the property **RK2** in Section 2.4. □

Remark 5.9. Parts (ii) and (iii) of Proposition 5.7 generalize Wolpert's computation of the $(1,1)$-form Θ for finite-dimensional Teichmüller spaces (see Theorem 5.5 and formula (5.3) in [Wol86]).

5.3. **Mumford-Morita-Miller characteristic forms.** In this section we consider $\pi:\mathcal{T}(1)\to T(1)$ as holomorphic fibration of the Hilbert manifolds and evaluate the Mumford-Morita-Miller forms κ_n on $T(1)$.

Theorem 5.10. *The characteristic forms κ_n are right-invariant on the Hilbert manifold $T(1)$ and for $\mu_1,\dots,\mu_n,\nu_1,\dots,\nu_n\in H^{-1,1}(\mathbb{D}^*)\simeq T_0T(1)$*

$$\begin{aligned}&\kappa_n(\mu_1,\dots,\mu_n,\bar{\nu}_1,\dots,\bar{\nu}_n)\\ =&\frac{i^n(n+1)!}{(2\pi)^{n+1}}\sum_{\sigma\in S_n}sgn(\sigma)\iint_{\mathbb{D}^*}G\left(\mu_1\bar{\nu}_{\sigma(1)}\right)\dots G\left(\mu_n\bar{\nu}_{\sigma(n)}\right)\rho(z)d^2z,\end{aligned}$$

where the sum goes over the permutation group S_n on n elements and $sgn(\sigma)$ is the sign of the permutation σ.

Proof. It is straightforward computation of the integral

$$\kappa_n(\mu_1,\dots,\mu_n,\bar{\nu}_1,\dots,\bar{\nu}_n) = \left(\frac{-i}{2\pi}\right)^{n+1} \iint\limits_{\mathbb{D}^*} \Theta^{n+1}\left(\partial_z,\partial_{\bar{z}},\tau_{\mu_1},\tau_{\bar{\nu}_1},\dots,\tau_{\mu_n},\tau_{\bar{\nu}_n}\right) dz\wedge d\bar{z}$$

using Part (ii) of Proposition 5.7. We need only to verify that the integral is convergent. This follows from the properties of the resolvent kernel in Section 2.4. Indeed, the property **RK3** assures that $G(\mu\bar{\nu})$ is bounded on $\mathbb{D}^*$ for $\mu,\nu\in\Omega^{-1,1}(\mathbb{D}^*)$, and properties **RK2** and **RK4** imply that for $\mu,\nu\in H^{-1,1}(\mathbb{D}^*)$,

$$\begin{aligned}\left|\iint\limits_{\mathbb{D}^*} G(\mu\bar{\nu})\rho(z)d^2z\right| &\leq \iint\limits_{\mathbb{D}^*}\iint\limits_{\mathbb{D}^*} G(z,u)|\mu(u)\nu(u)|\rho(u)\rho(z)d^2u d^2z \\ &= \iint\limits_{\mathbb{D}^*}\iint\limits_{\mathbb{D}^*} G(z,u)|\mu(u)\nu(u)|\rho(z)\rho(u)d^2z d^2u \\ &= \iint\limits_{\mathbb{D}^*} |\mu(u)\nu(u)|\rho(u)d^2u < \infty.\end{aligned}$$

□

Corollary 5.11. *On the Hilbert manifold* $T(1)$

$$\kappa_1 = \tfrac{1}{\pi^2}\,\omega_{WP}.$$

Proof. We have, using again the property **RK4** in Section 2.4,

$$\begin{aligned}\kappa_1(\mu,\bar{\nu}) =& \frac{i}{2\pi^2}\iint\limits_{\mathbb{D}^*} G(\mu\bar{\nu})\rho(z)d^2z = \frac{i}{2\pi^2}\iint\limits_{\mathbb{D}^*} \mu(z)\overline{\nu(z)}\rho(z)d^2z \\ =& \frac{1}{\pi^2}\,\omega_{WP}(\mu,\bar{\nu}).\end{aligned}$$

□

Remark 5.12. Combining Corollary 5.11, Part (ii) of the Proposition 5.7 and Theorem 5.3, we get another proof of Theorem 4.3 in [Teo02].

Remark 5.13. Theorem 5.10 generalizes Wolpert's result for finite-dimensional Teichmüller spaces (see Lemma 5.9 and Lemma 5.10 in [Wol86]) to the universal Teichmüller space.

6. First and second variations of the hyperbolic metric

Here we present a concise formula for the second variation of the hyperbolic metric in terms of the resolvent kernel. We are using the model $\mathbb{H}^2\simeq\mathbb{U}$, so that the density of the hyperbolic metric $\rho(z)=y^{-2}$ is a $(1,1)$ – tensor on $\mathbb{U}$.

6.1. **The first variation.** It is a classical result of Ahlfors [Ahl61] that the first variation of the hyperbolic metric at the origin of $T(1)$ is identically zero.

Lemma 6.1. *For every* $\mu \in \Omega^{-1,1}(\mathbb{U})$,

$$L_\mu \rho = 0.$$

Proof. Since

$$\rho^{\varepsilon\mu} = w_{\varepsilon\mu}^*(\rho) = -\frac{4\,|(w_{\varepsilon\mu})_z(z)|^2}{(w_{\varepsilon\mu}(z) - \overline{w_{\varepsilon\mu}(z)})^2},$$

we have

$$L_\mu \rho(z) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \rho^{\varepsilon\mu}(z) = -4\frac{F_z(z) + \overline{\Phi'(z)}}{(z-\bar{z})^2} + 8\frac{F(z) - \overline{\Phi(z)}}{(z-\bar{z})^3},$$

where $F = F[\mu]$, $\Phi = \Phi[\mu]$, and the result follows from Lemma 2.13. □

Remark 6.2. For the case $\mu \in \Omega^{-1,1}(\mathbb{U}, \Gamma)$, where Γ is a cofinite Fuchsian group, another proof of the Ahlfors result was given by Wolpert [Wol86].

6.2. **The second variation.** Set

$$L_\mu L_{\bar{\mu}} \rho = \frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\Big|_{\varepsilon=0} \rho^{\varepsilon\mu}.$$

We have

Proposition 6.3. *For every* $\mu \in \Omega^{-1,1}(\mathbb{U})$,

$$L_\mu L_{\bar{\mu}} \rho = \rho G(|\mu|^2).$$

Proof. Using the representation

$$\rho^{\varepsilon\mu}(z) = -4K_{\varepsilon\mu}(z, \bar{z}) \tag{6.1}$$

and the first formula in (2.24) we get

$$\frac{\partial}{\partial\varepsilon}\rho^{\varepsilon\mu}(z) = \frac{4}{\pi}\iint\limits_{\mathbb{U}} \mu(u) K_{\varepsilon\mu}(z, u) K_{\varepsilon\mu}(u, \bar{z}) d^2u, \tag{6.2}$$

where the integral is understood in the principal value sense. Setting $\varepsilon = 0$ in (6.2) and using Lemma 6.1, we obtain

$$\iint\limits_{\mathbb{U}} \frac{\mu(u)}{(u-z)^2(u-\bar{z})^2} d^2u = 0 \ \text{ for all } \ z \in \mathbb{U}. \tag{6.3}$$

Using formulas (2.23) and (2.24), we get from (6.2) the following integral representation for the second variation of the hyperbolic metric

$$\begin{aligned} L_\mu L_{\bar{\mu}} \rho(z) = -\frac{4}{\pi^2} \iint\limits_{\mathbb{U}} \iint\limits_{\mathbb{U}} \mu(u)\overline{\mu(v)} \Bigg(&\frac{1}{(u-\bar{v})^2(z-\bar{v})^2(u-\bar{z})^2} \\ &+ \frac{1}{(u-z)^2(u-\bar{v})^2(\bar{z}-\bar{v})^2} \Bigg) d^2u d^2v. \end{aligned} \tag{6.4}$$

The differentiation under the integral sign is justified by the same argument as in [Ahl62]. We transform the principal value integrals in (6.4) into the ordinary integrals by using the identity

$$\frac{1}{u-\bar{v}}=\frac{z-\bar{z}}{(u-\bar{z})(z-\bar{v})}+\frac{(\bar{v}-\bar{z})(z-u)}{(u-\bar{z})(z-\bar{v})(u-\bar{v})}, \tag{6.5}$$

which gives

$$\begin{aligned}&\frac{1}{(u-z)^2(u-\bar{v})^2(\bar{z}-\bar{v})^2}\\ &=\frac{1}{(u-\bar{z})^2(z-\bar{v})^2(u-\bar{v})^2}+\frac{2(z-\bar{z})}{(u-\bar{z})^3(z-\bar{v})^3(u-\bar{v})}\\ &+\frac{(z-\bar{z})^2}{(u-z)^2(u-\bar{z})^2(z-\bar{v})^2(\bar{z}-\bar{v})^2}+\frac{2(z-\bar{z})^2}{(u-z)(u-\bar{z})^3(z-\bar{v})^3(\bar{z}-\bar{v})}.\end{aligned}$$

Using (6.3) and Corollary 2.15 we see that last two terms in this formula do not contribute into the representation (6.4) and we obtain

$$\begin{aligned}L_\mu L_{\bar{\mu}}\rho(z)=-\frac{8}{\pi^2}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\mu(u)\overline{\mu(v)}\Bigg(&\frac{1}{(u-\bar{v})^2(z-\bar{v})^2(u-\bar{z})^2}\\ &+\frac{(z-\bar{z})}{(u-\bar{z})^3(u-\bar{v})(z-\bar{v})^3}\Bigg)d^2u d^2v.\end{aligned}$$

Now we apply the operator $2(\Delta_0+\frac{1}{2})$ to the bounded function $\rho^{-1}L_\mu L_{\bar{\mu}}\rho$ on $\mathbb{U}$. Using (6.5) it is straightforward to compute that

$$\begin{aligned}(2\Delta_0+1)\left(\frac{(z-\bar{z})^2}{(u-\bar{v})^2(z-\bar{v})^2(u-\bar{z})^2}+\frac{(z-\bar{z})^3}{(u-\bar{z})^3(u-\bar{v})(z-\bar{v})^3}\right)&\\ =\frac{9}{2}\frac{(z-\bar{z})^4}{(u-\bar{z})^4(z-\bar{v})^4}&,\end{aligned}$$

which, together with (2.22), gives

$$\begin{aligned}&(2\Delta_0+1)\left(\rho^{-1}L_\mu L_{\bar{\mu}}\rho\right)(z)\\ &=\frac{9}{\pi^2}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\mu(u)\overline{\mu(v)}\frac{(z-\bar{z})^4}{(u-\bar{z})^4(z-\bar{v})^4}d^2u d^2v=|\mu(z)|^2.\end{aligned}$$

Using the property **RK3** in Section 2.4 completes the proof. □

Corollary 6.4. *For every* $\mu,\nu\in\Omega^{-1,1}(\mathbb{D}^*)$,

$$\begin{aligned}G(\mu\bar{\nu})(z)=\frac{2}{\pi^2}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\mu(u)\overline{\nu(v)}\Bigg(&\frac{(z-\bar{z})^2}{(u-\bar{v})^2(z-\bar{v})^2(u-\bar{z})^2}\\ &+\frac{(z-\bar{z})^3}{(u-\bar{z})^3(u-\bar{v})(z-\bar{v})^3}\Bigg)d^2u d^2v.\end{aligned}$$

Remark 6.5. It follows from Proposition 6.3 by polarization that

$$L_\mu L_{\bar{\nu}}\rho = \left.\frac{\partial^2}{\partial\varepsilon_1\partial\bar{\varepsilon}_2}\right|_{\varepsilon_1=\varepsilon_2=0} \rho^{\varepsilon_1\mu+\varepsilon_2\nu} = \rho G(\mu\bar{\nu}).$$

Remark 6.6. For the case $\mu \in \Omega^{-1,1}(\mathbb{U},\Gamma)$, where Γ is a cofinite Fuchsian group, the formula for the second variation of the hyperbolic metric in Proposition 6.3 was first proved by Wolpert [Wol86]. However, method in [Wol86] does not work for the universal Teichmüller space $T(1)$. The proof of Proposition 6.3 shows that the Ahlfors' original singular integral representation (6.4) can be easily transformed to the closed form using the resolvent kernel.

7. Riemann curvature tensor

In this section we consider $T(1)$ as a Hilbert manifold equipped with the Weil-Petersson metric. We prove that the Weil-Petersson metric is Kähler , compute its Riemann and Ricci tensors, and show that the Ricci, holomorphic, and sectional curvatures are all negative. Since the Weil-Petersson metric is right-invariant, it is sufficient to compute these tensors at the origin of $T(1)$.

7.1. **The first variation of the Weil-Petersson metric.** For $\mu,\nu \in \Omega^{-1,1}(\mathbb{D}^*)$ set

$$Q(\mu,\nu) = P(R(\mu,\nu))\circ w_\nu\,\frac{(\overline{w_\nu})_{\bar{z}}}{(w_\nu)_z}.$$

Proposition 7.1. *For $\mu,\nu\in\Omega^{-1,1}(\mathbb{D})$,*

$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon}Q(\mu,\varepsilon\nu)\Big|_{\varepsilon=0} = 0,\\
&\frac{\partial}{\partial\bar{\varepsilon}}Q(\mu,\varepsilon\nu)\Big|_{\varepsilon=0} = -2\frac{\partial}{\partial\bar{z}}\rho^{-1}\frac{\partial}{\partial\bar{z}}G(\mu\bar{\nu}).
\end{aligned}$$

Proof. We will be using canonical complex anti-linear isomorphism $\Omega^{-1,1}(\mathbb{D}^*) \simeq \Omega^{-1,1}(\mathbb{D})$, given by the reflection (2.1), and the model $\mathbb{H}^2\simeq\mathbb{U}$ of the hyperbolic plane. From (2.6) we get

$$\rho^{\varepsilon\nu}(z)\,Q(\mu,\varepsilon\nu)(z) = \frac{12}{\pi}\iint\limits_{\mathbb{U}}\mu(u)K_{\varepsilon\nu}(u,\bar{z})^2d^2u. \tag{7.1}$$

It follows from equations (2.24) that

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}Q(\mu,\varepsilon\nu)(z) = \frac{6(z-\bar{z})^2}{\pi^2}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\mu(u)\nu(v)}{(u-\bar{z})^2(u-v)^2(v-\bar{z})^2}d^2ud^2v \tag{7.2}$$

and
(7.3)
$$\left.\frac{\partial}{\partial\bar\varepsilon}\right|_{\varepsilon=0} Q(\mu,\varepsilon\nu)(z)=\frac{6(z-\bar z)^2}{\pi^2}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\mu(u)\overline{\nu(v)}}{(u-\bar z)^2(u-\bar v)^2(\bar v-\bar z)^2}d^2ud^2v.$$
The integrals are understood in the principal value sense and differentiation under the integral sign in (7.1) is justified as in [Ahl62].

To prove that the integral (7.2) is zero, we use the identity
$$\frac{1}{(u-v)(u-\bar z)(v-\bar z)}=\frac{1}{(u-v)^2}\left(\frac{1}{v-\bar z}-\frac{1}{u-\bar z}\right),$$
and rewrite the integral (7.2) as follows
$$\begin{aligned}(7.4)\qquad &\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\mu(u)\nu(v)}{(u-\bar z)^2(u-v)^2(v-\bar z)^2}d^2ud^2v\\ &=\lim_{\varepsilon\to0}\iiiint\limits_{\mathbb{U}\times\mathbb{U}\setminus\{|u-v|<\varepsilon\}}\left(\frac{1}{(u-v)^4(v-\bar z)^2}-\frac{2}{(u-v)^5(v-\bar z)}\right.\\ &\left.+\frac{2}{(u-v)^5(u-\bar z)}+\frac{1}{(u-v)^4(u-\bar z)^2}\right)\mu(u)\nu(v)d^2ud^2v\\ &=I_1+I_2+I_3+I_4.\end{aligned}$$
Applying Lemma (2.12) to the principal value integrals over u in terms I_1, I_2 and I_4, we conclude that these terms vanish. Changing the order of integrations in I_3 (which is legitimate since domain of integration is invariant under the involution $(u,v)\mapsto(v,u)$) and applying Lemma (2.12) to the integral over v we conclude that the term I_3 also vanishes. This proves that the holomorphic variation of $Q(\mu,\nu)$ vanishes.

To prove the formula for the antiholomorphic variation, we again use the identity (6.5), which gives
$$\begin{aligned}\frac{1}{(u-\bar z)^2(u-\bar v)^2(\bar v-\bar z)^2}&=\frac{(u-z)^2}{(u-\bar z)^4(\bar v-z)^2(u-\bar v)^2}\\ &+\frac{(z-\bar z)^2}{(u-\bar z)^4(\bar v-\bar z)^2(\bar v-z)^2}+\frac{2(z-\bar z)^2(z-u)}{(u-\bar z)^5(z-\bar v)^3(\bar v-\bar z)}\\ &+\frac{2(z-\bar z)(z-u)^2}{(u-\bar z)^5(z-\bar v)^3(u-\bar v)}.\end{aligned}$$
Using formula (6.3) and Corollary 2.15 we see that the second and third terms do not contribute into (7.3), and we get
$$\begin{aligned}(7.5)\qquad \left.\frac{\partial}{\partial\bar\varepsilon}\right|_{\varepsilon=0} Q(\mu,\varepsilon\nu)(z)&=\frac{6(z-\bar z)^2}{\pi^2}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\mu(u)\overline{\nu(v)}\\ &\left(\frac{(u-z)^2}{(u-\bar z)^4(\bar v-z)^2(u-\bar v)^2}+\frac{2(z-\bar z)(z-u)^2}{(u-\bar z)^5(z-\bar v)^3(u-\bar v)}\right)d^2u^2v.\end{aligned}$$

Now applying $\partial_{\bar z}\rho^{-1}\partial_{\bar z}$ to the integral representation in Corollary 6.4 (the differentiation under the integral sign being trivially justified) we get the formula for the antiholomorphic variation. □

Set

$$\dot{Q}(\mu)[\nu] = \frac{\partial}{\partial\bar\varepsilon}\bigg|_{\varepsilon=0} Q(\mu, \varepsilon\nu).$$

Proposition 7.2. *Let* $\mu \in H^{-1,1}(\mathbb{D}^*)$ *and* $\nu \in \Omega^{-1,1}(\mathbb{D}^*)$. *Then* $\dot{Q}(\mu)[\nu] \in L^2(\mathbb{D}^*, \rho(z)d^2z)$ *and*

$$\left\|\dot{Q}(\mu)[\nu]\right\|_2^2 = \|\mu\bar\nu\|_2^2 - (\mu\bar\nu, G(\mu\bar\nu)),$$

where (,) *stands for the inner product in the Hilbert space* $L^2(\mathbb{D}^*, \rho(z)d^2z)$.

Proof. As in the proof of Proposition 7.1, it is convenient to use the isomorphism $\Omega^{-1,1}(\mathbb{D}^*) \simeq \Omega^{-1,1}(\mathbb{D})$. For $\mu \in BC^\infty(\mathbb{D}) \cap L^2(\mathbb{D}, \rho(z)d^2z)$ with compact support and $\nu \in BC^\infty(\mathbb{D})$ we set

$$\mathcal{Q}_\nu(\mu) = 2\frac{\partial}{\partial\bar z}\rho^{-1}\frac{\partial}{\partial\bar z}G(\mu\bar\nu).$$

We will prove that $\|\mathcal{Q}_\nu(\mu)\|_2^2 = \|\mu\bar\nu\|_2^2 - (\mu\bar\nu, G(\mu\bar\nu))$, so that $\mathcal{Q}_\nu$ extends to a bounded linear operator on $L^2(\mathbb{D}, \rho(z)d^2z)$. Since, according to Proposition 7.1, $\mathcal{Q}_\nu(\mu) = -\dot{Q}(\mu)[\nu]$ for $\mu \in H^{-1,1}(\mathbb{D}^*)$ and $\nu \in \Omega^{-1,1}(\mathbb{D}^*)$, the assertion follows from this fact.

From the explicit formula (2.18) we get the following estimates

$$\begin{aligned}(7.6)\qquad & G(z,w) = O((1-|z|^2)), \quad (\partial_z\rho^{-1}\partial_z)G(z,w) = O((1-|z|^2)),\\ & \partial_z G(z,w) = O(1), \quad \left(\partial_{\bar z}\left(\partial_z\rho^{-1}\partial_z\right)\right)G(z,w) = O(1) \quad \text{as} \quad |z|\to 1,\end{aligned}$$

uniformly on w on compact subsets of $\mathbb{D}$. Using the Stokes' theorem and the identity

$$\rho^{-1}\partial_z\,\rho^{-1}\partial_z\,\rho\,\partial_{\bar z}\,\rho^{-1}\partial_{\bar z} = \Delta_0(\Delta_0 + \tfrac{1}{2})$$

we get

$$\begin{aligned}\iint_{\mathbb{D}} |\mathcal{Q}_\nu(\mu)|^2\,\rho(z)d^2z &= 4\iint_{\mathbb{D}} \left(\rho^{-1}G_z(\bar\mu\nu)\right)_z\left(\rho^{-1}G_{\bar z}(\mu\bar\nu)\right)_{\bar z}\rho(z)d^2z\\ &= 4\iint_{\mathbb{D}} \Delta_0(\Delta_0+\tfrac{1}{2})G(\mu\bar\nu)G(\bar\mu\nu)\rho(z)d^2z\\ &= 2\iint_{\mathbb{D}} \Delta_0(\mu\bar\nu)G(\bar\mu\nu)\rho(z)d^2z,\end{aligned}$$

where in the last line we have used property **RK3** from Section 2.4. Due to the estimates (7.6) the boundary terms arising in the Stokes' formula vanish. Using the Stokes' theorem once again we finally get

$$\iint_{\mathbb{D}} |\mathcal{Q}_\nu(\mu)|^2 \rho(z) d^2 z = 2 \iint_{\mathbb{D}} \mu\bar{\nu}\Delta_0 G(\bar{\mu}\nu)\rho(z)d^2 z$$

$$= \iint_{\mathbb{D}} |\mu\bar{\nu}|^2 \rho(z) d^2 z - \iint_{\mathbb{D}} \mu\bar{\nu} G(\bar{\mu}\nu)\rho(z)d^2 z$$

$$= \|\mu\bar{\nu}\|_2^2 - (\mu\bar{\nu}, G(\mu\bar{\nu})).$$

The boundary terms again vanish due to (7.6) and Remark 3.2. □

Corollary 7.3. *For $\mu, \nu \in H^{-1,1}(\mathbb{D}^*)$ and $\kappa \in \Omega^{-1,1}(\mathbb{D}^*)$,*

$$\left(\dot{Q}(\mu)[\kappa], \dot{Q}(\nu)[\kappa]\right) = (\mu\bar{\kappa}, \nu\bar{\kappa}) - (\mu\bar{\kappa}, G(\nu\bar{\kappa}))\,.$$

Theorem 7.4. *For $\mu, \nu \in H^{-1,1}(\mathbb{D}^*)$ and $\kappa \in \Omega^{-1,1}(\mathbb{D}^*)$,*

$$\frac{\partial}{\partial\varepsilon} g_{\mu\bar{\nu}}(\varepsilon\kappa)\Big|_{\varepsilon=0} = 0.$$

Proof. Since $P^2 = P$, we get from (4.3),

(7.7)

$$g_{\mu\bar{\nu}}(\kappa) = \iint_{\mathbb{D}^*} Q(\mu,\kappa)\overline{Q(\nu,\kappa)}(1-|\kappa|^2) w_\kappa^*(\rho)(z) d^2 z = \iint_{\mathbb{D}^*} \mu \overline{Q(\nu,\kappa)} w_\kappa^*(\rho)(z) d^2 z,$$

so that

$$\frac{\partial}{\partial\varepsilon} g_{\mu\overline{\nu}}(\varepsilon\kappa)\Big|_{\varepsilon=0} = \left(\dot{Q}(\mu)[\kappa], \nu\right) + \left(\mu, \dot{Q}(\nu)[\kappa]\right) = \left(\mu, \dot{Q}(\nu)[\kappa]\right).$$

Differentiation under the integral sign is justified as in [Ahl62]. Thus for all $\mu, \nu \in H^{-1,1}(\mathbb{D}^*)$ and $\kappa \in \Omega^{-1,1}(\mathbb{D}^*)$

$$\left(\dot{Q}(\mu)[\kappa], \nu\right) = 0,$$

and the theorem follows. □

Let $\{\mu_n\}_{n=2}^\infty$ be an orthonormal basis for the Hilbert space $H^{-1,1}(\mathbb{D}^*)$,

$$\mu_n(z) = -\sqrt{\frac{n^3 - n}{8\pi}}(1 - |z|^2)^2 \bar{z}^{-n-2}, \quad n = 2, 3, \ldots,$$

and let $\{\varepsilon_n\}_{n=2}^\infty$ be corresponding Bers coordinates on the chart V_0. Since $\|\mu\|_2 = 2\|D_0\beta(\mu)\|_2$, it follows from Section 3.3 that $\sum_{n=2}^\infty |\varepsilon_n|^2 < \frac{4\pi}{3}$. Denote by $\frac{\partial}{\partial\varepsilon_n}$ corresponding directional derivatives — the vector field $\frac{\partial}{\partial\varepsilon_{\mu_n}}$ on V_0, and set $g_{m\bar{n}} = g_{\mu_m\bar{\mu}_n}$. Since the basis $\{\mu_n\}_{n=2}^\infty$ is orthonormal, $g_{m\bar{n}} = \delta_{mn}$ at the origin of $T(1)$.

Corollary 7.5. *The Weil-Petersson metric is a Kähler metric on the Hilbert manifold $T(1)$, and the Bers coordinates are geodesic coordinates at the origin of $T(1)$.*

Proof. It follows from Theorem 7.4 that

$$\frac{\partial g_{m\bar{n}}}{\partial \varepsilon_l}(0) = 0.$$

□

Remark 7.6. Propositions 7.1 and 7.2 and Theorem 7.4 generalize Wolpert's results for finite-dimensional Teichmüller spaces (see Lemma 2.7 and Theorem 2.9 in [Wol86]) to the universal Teichmüller space. In particular, our proof of Theorem 7.4 (after Proposition 7.2 has been established) is the same as in [Wol86].

7.2. **The second variation of the Weil-Petersson metric.** Due to Corollary 7.5, the Riemann tensor of the Weil-Petersson metric at the origin of $T(1)$ is given by

$$R_{k\bar{l}m\bar{n}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial \varepsilon_m \partial \bar{\varepsilon}_n}(0),$$

where we are using conventions of Yano and Bochner [YB53] in Hermitian geometry.

Theorem 7.7. *For $\mu, \nu \in H^{-1,1}(\mathbb{D}^*)$ and $\kappa \in \Omega^{-1,1}(\mathbb{D}^*)$,*

$$\left.\frac{\partial^2}{\partial \varepsilon \partial \bar{\varepsilon}} g_{\mu\bar{\nu}}(\varepsilon\kappa)\right|_{\varepsilon=0} = (\mu\bar{\kappa}, G(\nu\bar{\kappa})) + \left(\mu\bar{\nu}, G(|\kappa|^2)\right).$$

Proof. Differentiating representation (7.7) for $g_{\mu\bar{\nu}}(\varepsilon\kappa)$ with respect to ε and $\bar{\varepsilon}$ we get

$$\begin{aligned}
&\left.\frac{\partial^2}{\partial \varepsilon \partial \bar{\varepsilon}}\, g_{\mu\bar{\nu}}(\varepsilon\kappa)\right|_{\varepsilon=0} \\
&= \left(\left.\frac{\partial^2}{\partial \varepsilon \partial \bar{\varepsilon}}\, Q(\mu, \varepsilon\kappa)\right|_{\varepsilon=0}, \nu\right) + \left(\mu\bar{\nu}, \rho^{-1}\left.\frac{\partial^2}{\partial \varepsilon \partial \bar{\varepsilon}}\, \rho^{\varepsilon\kappa}\right|_{\varepsilon=0}\right) \\
&= \left(\left.\frac{\partial^2}{\partial \varepsilon \partial \bar{\varepsilon}}\, Q(\mu, \varepsilon\kappa)\right|_{\varepsilon=0}, \nu\right) + \left(\left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} Q(\mu, \varepsilon\kappa), \left.\frac{\partial}{\partial \bar{\varepsilon}}\right|_{\varepsilon=0} Q(\nu, \varepsilon\kappa)\right) \\
&+ \left(\mu, \left.\frac{\partial^2}{\partial \varepsilon \partial \bar{\varepsilon}}\, Q(\nu, \varepsilon\kappa)\right|_{\varepsilon=0}\right) + \left(\mu\bar{\nu}, \rho^{-1}\left.\frac{\partial^2}{\partial \varepsilon \partial \bar{\varepsilon}}\, \rho^{\varepsilon\kappa}\right|_{\varepsilon=0}\right) - \left(\mu\bar{\nu}, |\kappa|^2\right).
\end{aligned}$$

The differentiation under the integral sign is justified as in [Ahl62], provided that all integrals above are absolutely convergent. This follows from Proposition 6.3, property **RK3** in Section 2.4, Proposition 7.2 and the following

Lemma 7.8. *For $\mu \in H^{-1,1}(\mathbb{D}^*)$ and $\nu \in \Omega^{-1,1}(\mathbb{D}^*)$,*

$$\left.\frac{\partial^2}{\partial \varepsilon \partial \bar{\varepsilon}}\, Q(\mu, \varepsilon\nu)\right|_{\varepsilon=0} \in L^2(\mathbb{D}^*, \rho(z) d^2 z).$$

We relegate the proof of the lemma to the Appendix. Now comparing the two expressions for the second variation of $g_{\mu\bar{\nu}}$ and using Corollary 7.3 we get

$$\begin{aligned}\left(\mu, \frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}Q(\nu,\varepsilon\kappa)\bigg|_{\varepsilon=0}\right) &= -\left(\frac{\partial}{\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0}Q(\mu,\varepsilon\kappa), \frac{\partial}{\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0}Q(\nu,\varepsilon\kappa)\right) + \left(\mu\bar{\nu}, |\kappa|^2\right) \\ &= \left(\mu\bar{\kappa}, G(\nu\bar{\kappa})\right).\end{aligned}$$

Using Proposition 6.3, we finally obtain

$$\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0} g_{\mu\bar{\nu}}(\varepsilon\kappa) = \left(\mu\bar{\kappa}, G(\nu\bar{\kappa})\right) + \left(\mu\bar{\nu}, G(|\kappa|^2)\right).$$

□

Corollary 7.9. *At the origin of* $T(1)$,

$$R_{k\bar{l}m\bar{n}} = -\left(\mu_k\bar{\mu}_l, G(\bar{\mu}_m\mu_n)\right) - \left(\bar{\mu}_l\mu_m, G(\bar{\mu}_k\mu_n)\right).$$

Proof. It follows from Theorem 7.7 by polarization that

$$\begin{aligned}R_{\kappa\bar{\lambda}\mu\bar{\nu}} &= -\frac{\partial^2}{\partial\varepsilon_1\partial\bar{\varepsilon}_2}\bigg|_{\varepsilon_1=\varepsilon_2=0} g_{\mu\bar{\nu}}(\varepsilon_1\kappa+\varepsilon_2\lambda) \\ &= -(\kappa\bar{\lambda}, G(\bar{\mu}\nu)) - (\bar{\lambda}\mu, G(\bar{\kappa}\nu)).\end{aligned}$$

□

Remark 7.10. For finite-dimensional Teichmüller spaces this result was proved by Wolpert [Wol86]. Except Lemma 7.8, our derivation is the same as in [Wol86].

7.3. **Ricci and sectional curvatures.** The Ricci tensor at the origin of $T(1)$ for the orthonormal basis $\{\mu_n\}_{n=2}^{\infty}$ of $H^{-1,1}(\mathbb{D}^*)$ is defined by the following series

$$\mathcal{R}_{k\bar{l}} = \sum_{n=2}^{\infty} R_{k\bar{n}n\bar{l}}.$$

Theorem 7.11. *The Ricci tensor at the origin of* $T(1)$ *is well-defined and is given by*

$$\mathcal{R}_{k\bar{l}} = -\frac{13}{12\pi}\delta_{kl}.$$

Proof. Set $\mu = \mu_k, \nu = \mu_l$, and $\mathcal{R}_{\mu\bar{\nu}} = \mathcal{R}_{k\bar{l}}$. We have

$$\mathcal{R}_{\mu\bar{\nu}} = -\frac{2}{\pi}\sum_{n=2}^{\infty}(n^3-n)\left(\iint\limits_{\mathbb{D}^*}\iint\limits_{\mathbb{D}^*} G(z,w)\mu(z)z^{n-2}\overline{\nu(w)}\bar{w}^{n-2}d^2wd^2z\right.$$
$$\left.+\iint\limits_{\mathbb{D}^*}\iint\limits_{\mathbb{D}^*} G(z,w)\bar{z}^{n-2}z^{n-2}\mu(w)\overline{\nu(w)}\frac{(1-|z|^2)^2}{(1-|w|^2)^2}d^2wd^2z\right)$$
$$= -\frac{12}{\pi}\iint\limits_{\mathbb{D}^*}\iint\limits_{\mathbb{D}^*} G(z,w)\frac{\mu(z)\overline{\nu(w)}}{(1-z\bar{w})^4}d^2wd^2z$$
$$-\frac{3}{4\pi}\iint\limits_{\mathbb{D}^*}\iint\limits_{\mathbb{D}^*} G(z,w)\mu(w)\overline{\nu(w)}\rho(z)\rho(w)d^2wd^2z$$
$$= I_1 + I_2.$$

For the second integral, we use property **RK4** in Section 2.4 and get

$$I_2 = -\frac{3}{4\pi}\iint\limits_{\mathbb{D}}\mu(w)\overline{\nu(w)}\rho(w)d^2w = -\frac{3}{4\pi}g_{\mu\bar{\nu}}.$$

For the first integral using projection formula (2.7) we have

$$I_1 = -\frac{36}{\pi^2}\iint\limits_{\mathbb{D}^*}\iint\limits_{\mathbb{D}^*}\iint\limits_{\mathbb{D}^*} G(z,w)\mu(z)\overline{\nu(v)}\frac{(1-|w|^2)^2}{(1-z\bar{w})^4(1-w\bar{v})^4}d^2vd^2wd^2z.$$

Let

$$B(z,v) = \iint\limits_{\mathbb{D}^*} G(z,w)\frac{(1-|w|^2)^2}{(1-w\bar{v})^4(1-z\bar{w})^4}d^2w.$$

The kernel $B(z,v)$ satisfies

$$B(z,v) = B\Big(\sigma z, \sigma v\Big)\sigma'(z)^2\overline{\sigma'(v)}^2 \quad \text{for all} \quad \sigma\in\mathrm{PSU}(1,1), \tag{7.8}$$

and

$$B(0,v) = \iint\limits_{\mathbb{D}^*}\left(\frac{1}{2\pi}\frac{1+|w|^2}{1-|w|^2}\log\frac{1}{|w|^2}-\frac{1}{\pi}\right)\frac{(1-|w|^2)^2}{(1-\bar{v}w)^4}d^2w = \frac{1}{9}.$$

Hence

$$B(z,v) = \frac{1}{9(1-z\bar{v})^4}$$

and

$$I_1 = -\frac{4}{\pi^2} \iint\limits_{\mathbb{D}^*} \iint\limits_{\mathbb{D}^*} \mu(z)\overline{\nu(v)} \frac{1}{(1-z\bar{v})^4} d^2z d^2w$$
$$= -\frac{1}{3\pi} \iint\limits_{\mathbb{D}^*} \mu(z)\overline{\nu(z)}\rho(z) d^2z = -\frac{1}{3\pi} g_{\mu\bar{\nu}}.$$

Therefore.

$$\mathcal{R}_{\mu\bar{\nu}} = -\left(\frac{3}{4\pi} + \frac{1}{3\pi}\right) g_{\mu\bar{\nu}} = -\frac{13}{12\pi} g_{\mu\bar{\nu}}.$$

□

Since the Weil-Petersson metric on $T(1)$ is right-invariant, it follows from Theorem 7.11 that the Ricci tensor is well-defined everywhere on $T(1)$. Denote by Ric_{WP} corresponding Ricci $(1,1)$-form on $T(1)$. In terms of Bers coordinates $\{\varepsilon_n\}_{n=2}^{\infty}$ on the coordinate chart V_μ the Ricci form is given by

$$Ric_{WP} = \frac{i}{2} \sum_{k,l=2}^{\infty} \mathcal{R}_{k\bar{l}}\, d\varepsilon_k \wedge d\bar{\varepsilon}_l.$$

Corollary 7.12. *The universal Teichmüller space $T(1)$ is a Kähler-Einstein manifold with negative constant Ricci curvature,*

$$Ric_{WP} = -\frac{13}{12\pi}\, \omega_{WP}.$$

Proof. Since $(1,1)-$ forms ω_{WP} and Ric are right-invariant, the result immediately follows from Theorem 7.11 □

Remark 7.13. For the dense submanifold $\text{Möb}(S^1)\backslash \text{Diff}_+(S^1)$ of $T(1)$ the statement of Theorem 7.11 was established by different methods in [KY87] and [BR87a, BR87b]. The "magic ratio" $\frac{13}{12\pi}$ is omnipresent in the mathematics related to the string theory.

Let $\frac{\partial}{\partial t_\mu}, \frac{\partial}{\partial t_\nu} \in T_0^{\mathbb{R}} T(1)$ be real tangent vectors. According to [YB53], the sectional curvature of the section spanned by these vectors is given by R/g, where

$$R = R_{\mu\bar{\nu}\nu\bar{\mu}} + R_{\nu\bar{\mu}\mu\bar{\nu}} - R_{\mu\bar{\nu}\mu\bar{\nu}} - R_{\nu\bar{\mu}\nu\bar{\mu}},$$
$$g = 4 g_{\mu\bar{\mu}} g_{\nu\bar{\nu}} - 2|g_{\mu\bar{\nu}}|^2 - 2\,\text{Re}(g_{\mu\bar{\nu}})^2. \tag{7.9}$$

Similarly, the holomorphic sectional curvature of the section spanned by the holomorphic tangent vector $\frac{\partial}{\partial \varepsilon_\mu}$, where $g_{\mu\bar{\mu}} = 1$, is given by $R_{\mu\bar{\mu}\mu\bar{\mu}}$.

As in the finite-dimensional case [Wol86], we have

Theorem 7.14. *Sectional and holomorphic sectional curvatures of $T(1)$ are negative.*

Proof. For a section spanned by $\frac{\partial}{\partial \varepsilon_\mu}$, the holomorphic sectional curvature is obviously negative: $\mu \neq 0$ so that $G(|\mu|^2) > 0$, and $(|\mu|^2, G(|\mu|^2)) > 0$.

For a section spanned by the real tangent vectors $\frac{\partial}{\partial t_\mu}$ and $\frac{\partial}{\partial t_\nu}$, using Cauchy-Schwarz inequality, it is easy to see that g is positive. Using Corollary 7.9, we get

$$R = (\mu\bar{\nu}, G(\bar{\mu}\nu)) + (\bar{\mu}\nu, G(\mu\bar{\nu})) - (|\mu|^2, G(|\nu|^2)) - (|\nu|^2, G(|\mu|^2)).$$

From the property **RK2** and Cauchy-Schwarz inequality we have

$$|G(\mu\bar{\nu})(z)| \leq \iint_{\mathbb{D}^*} G(z,w)^{1/2}|\mu(w)|G(z,w)^{1/2}|\nu(w)|\rho(w)d^2w$$

$$\leq \left(\iint_{\mathbb{D}^*} G(z,w)|\mu(w)|^2\rho(w)d^2w\right)^{1/2}\left(\iint_{\mathbb{D}^*} G(z,w)|\nu(w)|^2\rho(w)d^2w\right)^{1/2}$$

so that

$$|(\bar{\mu}\nu, G(\mu\bar{\nu}))| \leq \iint_{\mathbb{D}^*} |\mu\nu| G(|\mu|^2)^{1/2}G(|\nu|^2)^{1/2}\rho(z)d^2z$$

$$\leq \left(\iint_{\mathbb{D}^*} |\mu|^2 G(|\nu|^2)\rho(z)d^2z\right)^{1/2}\left(\iint_{\mathbb{D}^*} |\nu|^2 G(|\mu|^2)\rho(z)d^2z\right)^{1/2}.$$

Hence R is negative by Cauchy-Schwarz inequality. □

8. Finite-dimensional Teichmüller spaces

Curvature properties of finite-dimensional Teichmüller spaces were extensively studied by Ahlfors [Ahl62], Royden [Roy75], and especially by Wolpert [Wol86]. Here, for the Teichmüller space $T(\Gamma)$ for a cofinite Fuchsian group Γ we show how to get Wolpert's explicit formulas from the curvature formulas for the Hilbert manifold $T(1)$, derived in Section 7.

First note that canonical embedding of a finite-dimensional complex manifold $T(\Gamma)$ into $T(1)$ is holomorphic with respect to the Banach manifold structure on $T(1)$ but not with respect to the Hilbert manifold structure on $T(1)$. Indeed, for a cofinite Fuchsian group Γ the finite-dimensional vector space $\Omega^{-1,1}(\mathbb{D}^*, \Gamma)$ is not a subspace of the Hilbert space $H^{-1,1}(\mathbb{D}^*)$, but rather

$$\Omega^{-1,1}(\mathbb{D}^*, \Gamma) \cap H^{-1,1}(\mathbb{D}^*) = \{0\}.$$

Thus the Weil-Petersson metric on $T(\Gamma)$, defined in Section 2.3, is not a pull-back of the Weil-Petersson metric on $T(1)$. However, due to Lemma 2.9 we can represent the Petersson inner product on the tangent space at the origin of $T(\Gamma)$ as an "average" of the inner products in $T(1)$. Namely, using canonical complex anti-linear isomorphisms $\Omega^{-1,1}(\mathbb{D}^*,\Gamma)\simeq\Omega^{-1,1}(\mathbb{D},\Gamma)$ and $H^{-1,1}(\mathbb{D}^*)\simeq H^{-1,1}(\mathbb{D})$, we have

$$\begin{aligned}\langle\mu,\nu\rangle_{WP}=\iint\limits_{\Gamma\backslash\mathbb{D}}\mu\bar{\nu}\rho(z)d^2z&=\lim_{r\to1^-}\frac{A(\Gamma\backslash\mathbb{D})}{A(\mathbb{D}_r)}\iint\limits_{\mathbb{D}_r}\mu\bar{\nu}\rho(z)d^2z\\&=\lim_{r\to1^-}\frac{A(\Gamma\backslash\mathbb{D})}{A(\mathbb{D}_r)}\iint\limits_{\mathbb{D}}\mu_r\bar{\nu}_r\rho(z)d^2z.\end{aligned}$$

Here $\mu,\nu\in\Omega^{-1,1}(\mathbb{D},\Gamma)$ and $\mu_r=\chi_r\mu,\nu_r=\chi_r\nu$, where χ_r is the characteristic function of $\mathbb{D}_r=\{z\in\mathbb{D}:|z|\leq r\}$. In what follows we will denote by $(\ ,\ )_\Gamma$ the Petersson inner product $\langle\ ,\ \rangle_{WP}$ in $\Omega^{-1,1}(\mathbb{D})$, as well as the inner product for the Hilbert space $L^2(\Gamma\backslash\mathbb{D},\rho(z)d^2z)$. Since they are given by the same formula, there would be no confusion. Moreover, for $\mu\in\Omega^{-1,1}(\mathbb{D},\Gamma)$, $|\mu|\in L^2(\Gamma\backslash\mathbb{D},\rho(z)d^2z)$.

Lemma 8.1. *Let $\mu\in\Omega^{-1,1}(\mathbb{D})$ and $\nu\in L^\infty(\mathbb{D})\cap L^1(\mathbb{D},\rho(z)d^2z)$. Then*

(i) *For $0<r<1$,*

$$P(\mu_r)\in H^{-1,1}(\mathbb{D})\quad\textit{and}\quad P(\mu_r)(z)=O\left((1-|z|^2)^2\right)\quad\textit{as}\quad|z|\to1.$$

(ii)

$$\lim_{r\to1^-}\iint\limits_{\mathbb{D}}P(\mu_r)\bar{\nu}\,\rho(z)d^2z=\iint\limits_{\mathbb{D}}\mu\bar{\nu}\,\rho(z)d^2z.$$

Proof. Since

$$\iint\limits_{\mathbb{D}}|P(\mu_r)|^2\rho(z)d^2z=\iint\limits_{\mathbb{D}}P(\mu_r)\bar{\mu}_r\,\rho(z)d^2z=\iint\limits_{\mathbb{D}_r}P(\mu_r)\bar{\mu}_r\,\rho(z)d^2z<\infty,$$

$P(\mu_r)\in H^{-1,1}(\mathbb{D})$. Using $\mu(z)=-\frac{(1-|z|^2)^2}{2}\sum_{n=2}^\infty(n^3-n)a_n\bar{z}^{n-2}$ and (2.7), we get

$$P(\mu_r)(z)=-\frac{(1-|z|^2)^2}{4}\sum_{n=2}^\infty(n^3-n)a_n\left(\frac{r^{2n+2}}{n+1}-\frac{2r^{2n}}{n}+\frac{r^{2n-2}}{n-1}\right)\bar{z}^{n-2},$$

so that $(1-|z|^2)^{-2}P(\mu_r)(z)$ is continuous on $|z|=1$.

To prove part (ii), consider the estimate

$$\begin{aligned}|(\mu-P(\mu_r))(z)|\leq&\frac{3(1-|z|^2)^2}{\pi}\|\mu\|_\infty\iint\limits_{\mathbb{D}\backslash\mathbb{D}_r}\frac{d^2u}{|1-u\bar{z}|^4}\\=&3\|\mu\|_\infty(1-|z|^2)^2\sum_{n=1}^\infty n|z|^{2n-2}(1-r^{2n})\\=&3\|\mu\|_\infty\left(1-\frac{r^2(1-|z|^2)^2}{(1-r^2|z|^2)^2}\right).\end{aligned}$$

For fixed r the right hand side of this estimate is an increasing function of $|z|$, so that

$$\sup_{|z|\leq s} |(\mu - P(\mu_r))(z)| \leq 3\|\mu\|_\infty \left(1 - \frac{r^2(1-s^2)^2}{(1-r^2s^2)^2}\right),$$

and for fixed s,

$$\lim_{r\to 1^-} \sup_{|z|\leq s} |(\mu - P(\mu_r))(z)| = 0.$$

Also for fixed r we have the estimate

$$\|\mu - P(\mu_r)\|_\infty \leq 3\|\mu\|_\infty.$$

Now since $\nu \in L^1(\mathbb{D}, \rho(z)d^2z)$, for every $\varepsilon > 0$ there exists $0 < s < 1$ such that

$$\iint_{\mathbb{D}\setminus\mathbb{D}_s} |\nu|\,\rho(z)d^2z \leq \varepsilon,$$

and we obtain,

$$\left|\iint_{\mathbb{D}} (\mu - P(\mu_r))\,\bar\nu\rho(z)d^2z\right|$$

$$\leq \iint_{\mathbb{D}_s} |(\mu - P(\mu_r))|\,|\nu|\,\rho(z)d^2z + \iint_{\mathbb{D}\setminus\mathbb{D}_s} |(\mu - P(\mu_r))|\,|\nu|\,\rho(z)d^2z$$

$$\leq \sup_{|z|\leq s} |(\mu - P(\mu_r))(z)| \iint_{\mathbb{D}} |\nu|\,\rho(z)d^2z + 3\varepsilon\|\mu\|_\infty.$$

Passing to the limit $r \to 1^-$, we get

$$\lim_{r\to 1^-} \left|\iint_{\mathbb{D}} (\mu - P(\mu_r))\,\bar\nu\,\rho(z)d^2z\right| \leq 3\varepsilon\|\mu\|_\infty.$$

Since ε is arbitrary, the result follows. □

Lemma 8.2. *For $\mu, \nu \in \Omega^{-1,1}(\mathbb{D}, \Gamma)$,*

$$(\mu, \nu)_\Gamma = \lim_{r\to 1^-} \lim_{s\to 1^-} \frac{A(\Gamma\setminus\mathbb{D})}{A(\mathbb{D}_r)} \iint_{\mathbb{D}} P(\mu_s)\overline{P(\nu_r)}\rho(z)d^2z.$$

Proof. Since $\mu \in \Omega^{-1,1}(\mathbb{D}, \Gamma) \subset \Omega^{-1,1}(\mathbb{D})$,

$$\iint_{\mathbb{D}} \mu_r\bar\nu_r\rho(z)d^2z = \iint_{\mathbb{D}} \mu\bar\nu_r\rho(z)d^2z = \iint_{\mathbb{D}} \mu\overline{P(\nu_r)}\rho(z)d^2z.$$

According to part (i) of Lemma 8.1, $P(\nu_r) \in L^1(\mathbb{D}, \rho(z)d^2z)$ for $0 < r < 1$, so that the result follows from part (ii) of Lemma 8.1. □

Remark 8.3. The limits in Lemma 8.2 can not be interchanged. Indeed, it follows from part (ii) of Lemma 8.1 that for fixed $s < 1$ the limit $r \to 1$ is always zero.

In a neighborhood of the origin in $T(\Gamma)$ the Weil-Petersson metric is given by

$$g_{\mu\bar{\nu}}(\kappa) = \iint\limits_{\Gamma_\kappa \backslash \mathbb{D}} P(R(\mu,\kappa))\overline{P(R(\nu,\kappa))}\rho(z)d^2z,$$

where $\kappa \in \Omega^{-1,1}(\mathbb{D},\Gamma)$, $\|\kappa\|_\infty$ is sufficiently small, and $\Gamma_\kappa = w_\kappa \circ \Gamma \circ w_\kappa^{-1}$.

Lemma 8.4. *Let* $\mu,\nu \in \Omega^{-1,1}(\mathbb{D},\Gamma)$. *For* $\kappa \in \Omega^{-1,1}(\mathbb{D},\Gamma)$, $\|\kappa\|_\infty$ *sufficiently small,*

$$g_{\mu\bar{\nu}}(\kappa) = \lim_{r\to 1^-}\lim_{s\to 1^-}\frac{A(\Gamma\backslash\mathbb{D})}{A(\mathbb{D}_r)}\iint\limits_{\mathbb{D}} P(R(P(\mu_s),\kappa))\overline{P(R(P(\nu_r),\kappa))}\rho(z)d^2z.$$

Proof. First, we have

$$\begin{aligned}&\iint\limits_{\mathbb{D}} P(R(P(\mu_s),\kappa))\overline{P(R(P(\nu_r),\kappa))}\rho(z)d^2z\\ &=\frac{12}{\pi}\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}}\frac{P(\mu_s)(u)(w_\kappa)_u(u)^2\overline{P(\nu_r)(z)(w_\kappa)_z(z)^2}}{(1-w_\kappa(u)\overline{w_\kappa(z)})^4}d^2zd^2u.\end{aligned}$$

Since $\rho P(\nu_r)$ is bounded on $\mathbb{D}$, and for $\|\kappa\|_\infty$ sufficiently small $(1/2)\rho \leq w_\kappa^*\rho \leq (3/2)\rho$, we conclude that $\rho R(P(\nu_r,\kappa))$ is also bounded on $\mathbb{D}$. As the result,

$$\begin{aligned}&\iint\limits_{\mathbb{D}}\left|\iint\limits_{\mathbb{D}}\frac{(w_\kappa)_u(u)^2\overline{P(\nu_r)(z)(w_\kappa)_z(z)^2}}{(1-w_\kappa(u)\overline{w_\kappa(z)})^4}d^2z\right|d^2u\\ &=\iint\limits_{\mathbb{D}}\left|\iint\limits_{\mathbb{D}}\frac{\overline{R(P(\nu_r),\kappa)(z)}}{(1-u\bar{z})^4}d^2z\right|\frac{d^2u}{1-|\kappa(u)|^2}\\ &\leq C\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}}\frac{(1-|z|^2)^2}{|1-u\bar{z}|^4}d^2zd^2u = \pi^2C < \infty.\end{aligned}$$

It follows from part (ii) of Lemma 8.1 that

$$\lim_{s\to 1^-}\iint\limits_{\mathbb{D}} P(R(P(\mu_s),\kappa))\overline{P(R(P(\nu_r),\kappa))}\rho(z)d^2z$$
$$=\frac{12}{\pi}\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}}\frac{\mu(u)(w_\kappa)_u(u)^2\overline{P(\nu_r)(z)(w_\kappa)_z(z)^2}}{(1-w_\kappa(u)\overline{w_\kappa(z)})^4}d^2zd^2u$$
$$=\iint\limits_{\mathbb{D}} P(R(\mu,\kappa))\overline{P(R(P(\nu_r),\kappa))}\rho(z)d^2z.$$

Now

$$\iint\limits_{\mathbb{D}} P(R(\mu,\kappa))\overline{P(R(P(\nu_r),\kappa))}\rho(z)d^2z$$
$$=\frac{144}{\pi^2}\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}_r}\iint\limits_{\mathbb{D}}\frac{\mu(u)(w_\kappa)_u(u)^2\overline{\nu(v)(w_\kappa)_z(z)^2}}{(1-w_\kappa(u)\overline{w_\kappa(z)})^4(1-z\bar{v})^4}\rho(z)^{-1}d^2ud^2vd^2z$$
$$=\frac{144}{\pi^2}\iint\limits_{\mathbb{D}_r}\lambda(v)\overline{\nu(v)}\rho(v)d^2v,$$

where

$$\lambda(v)=\rho(v)^{-1}\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}}\frac{\mu(u)(w_\kappa)_u(u)^2\overline{(w_\kappa)_z(z)^2}\rho(z)^{-1}}{(1-w_\kappa(u)\overline{w_\kappa(z)})^4(1-z\bar{v})^4}d^2ud^2z.$$

Since $\mu\in\Omega^{-1,1}(\mathbb{D},\Gamma)$ and $w_\kappa\circ\Gamma\circ w_\kappa^{-1}=\Gamma_\kappa\subset\mathrm{PSU}(1,1)$, it is easy to see that $\lambda\in\Omega^{-1,1}(\mathbb{D},\Gamma)$. Using Lemma 2.9, we get

$$\lim_{r\to 1^-}\frac{A(\Gamma\backslash\mathbb{D})}{A(\mathbb{D}_r)}\iint\limits_{\mathbb{D}} P(R(\mu,\kappa))\overline{P(R(P(\nu_r),\kappa))}\rho(z)d^2z$$
$$=\frac{144}{\pi^2}\iint\limits_{\Gamma\backslash\mathbb{D}}\lambda(v)\overline{\nu(v)}\rho(v)d^2v$$
$$=\frac{144}{\pi^2}\iint\limits_{\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\mathbb{D}}\frac{\mu(u)(w_\kappa)_u(u)^2\overline{\nu(v)(w_\kappa)_z(z)^2}}{(1-w_\kappa(u)\overline{w_\kappa(z)})^4(1-z\bar{v})^4}\rho(z)^{-1}d^2ud^2vd^2z$$
$$=\frac{144}{\pi^2}\iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}}\frac{\mu(u)(w_\kappa)_u(u)^2\overline{\nu(v)(w_\kappa)_z(z)^2}}{(1-w_\kappa(u)\overline{w_\kappa(z)})^4(1-z\bar{v})^4}\rho(z)^{-1}d^2ud^2vd^2z$$
$$=\iint\limits_{\Gamma\backslash\mathbb{D}} P(R(\mu,\kappa))\overline{P(R(\nu,\kappa))}\rho(z)d^2z,$$

where we have used that the integrals above do not change if we let any one of the integrations to range over $\Gamma\backslash\mathbb{D}$ while others range over $\mathbb{D}$ (cf. [Ahl62]).

The latter property follows from the fact that μ,ν and κ are $(-1,1)$ tensors for Γ, and the representation $\mathbb{D}=\bigcup_{\gamma\in\Gamma}\gamma(\Gamma\backslash\mathbb{D})$. ☐

Theorem 8.5. *For $\mu,\nu,\kappa\in\Omega^{-1,1}(\mathbb{D},\Gamma)$,*

$$\begin{aligned}\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} g_{\mu\bar{\nu}}(\varepsilon\kappa) =&0,\\ \left.\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\right|_{\varepsilon=0} g_{\mu\bar{\nu}}(\varepsilon\kappa) =& (\mu\bar{\kappa},G_\Gamma(\nu\bar{\kappa}))_\Gamma+\left(\mu\bar{\nu},G_\Gamma(|\kappa|^2)\right)_\Gamma.\end{aligned}$$

Proof. We will use Lemma 8.4 and Theorems 7.4 and 7.7, provided one can interchange $\frac{\partial}{\partial\varepsilon}$, $\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}$ with the limits. This can be done as in [Ahl62] by showing that limits of corresponding derivatives converge uniformly on ε in a neighborhood of 0. We omit these standard arguments and concentrate on actual computations.

For the first variation of the Weil-Petersson metric we get

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} g_{\mu\bar{\nu}}(\varepsilon\kappa)=\lim_{r\to1^-}\lim_{s\to1^-}\frac{A(\Gamma\backslash\mathbb{D})}{A(\mathbb{D}_r)}\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} g_{P(\mu_s)\overline{P(\nu_r)}}(\varepsilon\kappa).$$

Since $P(\mu_s),P(\nu_r)\in H^{-1,1}(\mathbb{D})$, we conclude from Theorem 7.4 that this is identically zero.

Similarly, for the second variation we have

$$\left.\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\right|_{\varepsilon=0} g_{\mu\bar{\nu}}(\varepsilon\kappa)=\lim_{r\to1^-}\lim_{s\to1^-}\frac{A(\Gamma\backslash\mathbb{D})}{A(\mathbb{D}_r)}\left.\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\right|_{\varepsilon=0} g_{P(\mu_s)\overline{P(\nu_r)}}(\varepsilon\kappa).$$

Since $P(\nu_r),P(\mu_s)\in H^{-1,1}(\mathbb{D})$ and $\kappa\in\Omega^{-1,1}(\mathbb{D},\Gamma)\subset\Omega^{-1,1}(\mathbb{D})$, we get from Theorem 7.7[7],

$$\begin{aligned}&\left.\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\right|_{\varepsilon=0} g_{\mu\bar{\nu}}(\varepsilon\kappa)\\ &=\lim_{r\to1^-}\lim_{s\to1^-}\frac{A(\Gamma\backslash\mathbb{D})}{A(\mathbb{D}_r)}\left((P(\mu_s)\bar{\kappa},G(P(\nu_r)\bar{\kappa}))+\left(P(\mu_s)\overline{P(\nu_r)},G(|\kappa|^2)\right)\right).\end{aligned}$$

By properties **RK2** and **RK4** in Section 2.4,

$$\begin{aligned}\iint\limits_{\mathbb{D}}|G(P(\nu_r)\kappa)|\,|\kappa|\rho(z)d^2z\leq&\|\kappa\|_\infty^2\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}}G(z,w)\,|P(\nu_r)(w)|\,\rho(z)\rho(w)d^2wd^2z\\ =&\|\kappa\|_\infty^2\iint\limits_{\mathbb{D}}|P(\nu_r)(w)|\,\rho(w)d^2w<\infty,\end{aligned}$$

and by property **RK3** $G(|\kappa|^2)$ is bounded on $\mathbb{D}$, so that it follows from Lemma 8.1 that

$$\left.\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\right|_{\varepsilon=0} g_{\mu\bar{\nu}}(\varepsilon\kappa)=\lim_{r\to1^-}\frac{A(\Gamma\backslash\mathbb{D})}{A(\mathbb{D}_r)}\left((\mu\bar{\kappa},G(P(\nu_r)\bar{\kappa}))+\left(\mu\overline{P(\nu_r)},G(|\kappa|^2)\right)\right).$$

[7]It is for this case that we need the condition $\kappa\in\Omega^{-1,1}(\mathbb{D})$ in Theorem 7.7.

We have

$$\left(\mu\bar{\kappa}, G(P(\nu_r)\bar{\kappa})\right) = \iint\limits_{\mathbb{D}_r} \lambda_1(v)\overline{\nu(v)}\rho(v)d^2v,$$

$$\left(\mu\overline{P(\nu_r)}, G(|\kappa|^2)\right) = \iint\limits_{\mathbb{D}_r} \lambda_2(v)\overline{\nu(v)}\rho(v)d^2v,$$

where

$$\lambda_1(v) = \frac{12}{\pi}\rho(v)^{-1}\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}} \frac{\mu(z)\overline{\kappa(z)}G(z,u)\kappa(u)}{(1-u\bar{v})^4}\rho(z)d^2ud^2z,$$

$$\lambda_2(v) = \frac{12}{\pi}\rho(v)^{-1}\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}} \frac{\mu(z)G(z,u)|\kappa|^2(u)}{(1-z\bar{v})^4}\rho(u)d^2ud^2z,$$

and $\lambda_1, \lambda_2 \in \Omega^{-1,1}(\mathbb{D}, \Gamma)$. It follows from Lemma 2.9 that

$$\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0} g_{\mu\bar{\nu}}(\varepsilon\kappa) = \iint\limits_{\Gamma\backslash\mathbb{D}} \lambda_1(v)\overline{\nu(v)}\rho(v)d^2v + \iint\limits_{\Gamma\backslash\mathbb{D}} \lambda_2(v)\overline{\nu(v)}\rho(v)d^2v$$

$$= \frac{12}{\pi}\iint\limits_{\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\mathbb{D}} \mu(z)\overline{\kappa(z)}G(z,u)\kappa(u)\frac{\overline{\nu(v)}}{(1-u\bar{v})^4}\rho(z)d^2ud^2vd^2z$$

$$+\frac{12}{\pi}\iint\limits_{\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\mathbb{D}} \mu(z)G(z,u)|\kappa(u)|^2\frac{\overline{\nu(v)}}{(1-z\bar{v})^4}\rho(u)d^2ud^2vd^2z$$

$$= I_1 + I_2.$$

As before, the integrals above do not change if we let any one of the integrations to range over $\Gamma\backslash\mathbb{D}$ while others range over $\mathbb{D}$. We have, using property **RK1** , (2.7) and (2.19),

$$I_1 = \frac{12}{\pi}\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}} \mu(z)\overline{\kappa(z)}G(z,u)\kappa(u)\frac{\overline{\nu(v)}}{(1-u\bar{v})^4}\rho(z)d^2ud^2vd^2z$$

$$= \iint\limits_{\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}} \mu(z)\overline{\kappa(z)}G(z,u)\kappa(u)\overline{\nu(u)}\rho(u)\rho(z)d^2ud^2z$$

$$= \iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}} \mu(z)\overline{\kappa(z)}G_\Gamma(z,u)\kappa(u)\overline{\nu(u)}\rho(u)\rho(z)d^2ud^2z$$

$$= \left(\mu\bar{\kappa}, G_\Gamma(\nu\bar{\kappa})\right)_\Gamma.$$

Similarly,

$$\begin{aligned} I_2 &= \iint\limits_{\mathbb{D}} \iint\limits_{\Gamma\backslash\mathbb{D}} \mu(z)\overline{\nu(z)}G(z,u)\left|\kappa(u)\right|^2\rho(u)\rho(z)d^2ud^2z \\ &= \iint\limits_{\Gamma\backslash\mathbb{D}} \iint\limits_{\Gamma\backslash\mathbb{D}} \mu(z)\overline{\nu(z)}G_\Gamma(z,u)\left|\kappa(u)\right|^2\rho(u)\rho(z)d^2ud^2z \\ &= \left(\mu\bar{\nu}, G_\Gamma(|\kappa|^2)\right)_\Gamma, \end{aligned}$$

and the assertion follows. □

Remark 8.6. Theorem 8.5 was proved by Wolpert [Wol86], and all results on Ricci, sectional, and scalar curvatures for finite-dimensional Teichmüller spaces follow from it.

We conclude this section by deriving a formula for Ricci tensor different from [Wol86], and indicating its application. Let $\mu_1,\dots,\mu_d$ be an orthonormal basis of $\Omega^{-1,1}(\mathbb{D},\Gamma)$, which is a subspace of the Hilbert space $L^2(\mathbb{D},\Gamma)$ of Beltrami differentials μ for Γ such that $|\mu|\in L^2(\Gamma\backslash\mathbb{D},\rho(z)d^2z)$. Let $P: L^2(\mathbb{D},\Gamma)\to\Omega^{-1,1}(\mathbb{D},\Gamma)$ be the orthogonal projector. It follows from the definition and representation (2.7) that P is an integral operator with kernel

$$P(z,w)=\sum_{n=1}^{d}\mu_n(z)\overline{\mu_n(w)}=\frac{12}{\pi}\rho(z)^{-1}\rho(w)^{-1}\sum_{\gamma\in\Gamma}\frac{\gamma'(w)^2}{(1-\bar{z}\gamma(w))^4}.$$

The Ricci tensor at the origin of $T(\Gamma)$ is given by

$$\begin{aligned} \text{(8.1)}\quad \mathcal{R}_{\mu\bar{\nu}} &= \sum_{n=1}^{d} R_{\mu\bar{\mu}_n\mu_n\bar{\nu}} = -\sum_{n=1}^{d}\left(\left(\mu\bar{\nu}, G_\Gamma(|\mu_n|^2)\right)_\Gamma + \left(\mu\bar{\mu}_n, G_\Gamma(\nu\bar{\mu}_n)\right)_\Gamma\right) \\ &= -\sum_{n=1}^{d}\left(\iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}} \mu(z)\overline{\nu(z)}G_\Gamma(z,w)|\mu_n(w)|^2\rho(w)\rho(z)d^2wd^2z\right. \\ &\qquad \left. + \iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}} \mu(z)\overline{\mu_n(z)}G_\Gamma(z,w)\mu_n(w)\overline{\nu(w)}\rho(w)\rho(z)d^2wd^2z\right) \\ &= -\frac{12}{\pi}\iint\limits_{\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}} \mu(z)\overline{\nu(z)}G(z,w)\sum_{\gamma\in\Gamma}\frac{\gamma'(w)^2}{(1-\bar{w}\gamma(w))^4}\rho(w)^{-1}\rho(z)d^2wd^2z \\ &\qquad -\frac{12}{\pi}\iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\mathbb{D}} \mu(z)\overline{\nu(w)}G(z,w)\sum_{\gamma\in\Gamma}\frac{\gamma'(z)^2}{(1-\bar{w}\gamma(z))^4}d^2wd^2z, \end{aligned}$$

where nothing changes if we let any of the integrations to range over $\Gamma\backslash\mathbb{D}$ while other range over D.

It is instructive to compare the Ricci curvatures of the finite-dimensional Teichmüller space $T(\Gamma)$ and that of the universal Teichmüller space $T(1)$.

First, $T(\Gamma)$ is no longer a Kähler -Einstein manifold. Second, the sum over Γ in (8.1) can be transformed into the sum over the conjugacy classes of Γ. As is in [TZ91], using variational formulas for the Selberg zeta-function, we find that the contribution of hyperbolic conjugacy classes is the second variation of the Selberg zeta-function at $s = 2$. The contribution of parabolic conjugacy classes (if they are present) yields a new Kähler metric on $T(\Gamma)$, introduced in [TZ91]. The contribution of the identity element, as it follows from Theorem 7.11, is

$$-\frac{3}{4\pi}\iint\limits_{\mathbb{D}}\iint\limits_{\Gamma\backslash\mathbb{D}}\mu(z)\overline{\nu(z)}G(z,w)\rho(w)\rho(z)d^2wd^2z$$

$$-\frac{12}{\pi}\iint\limits_{\Gamma\backslash\mathbb{D}}\iint\limits_{\mathbb{D}}\mu(z)\overline{\nu(w)}G(z,w)\frac{1}{(1-z\bar{w})^4}d^2wd^2z=-\frac{13}{12\pi}(\mu,\nu)_{\Gamma}.$$

As the result, we obtain a local index theorem for families of $\bar{\partial}$-operators acting on quadratic differentials on Riemann surfaces, proved in [TZ91]. The above arguments interpret it as an "averaged form" of Theorem 7.11. Detailed derivation of the local index theorem for families from (8.1) will be presented elsewhere.

Appendix

Here we prove Lemma 7.8. Using the model $\mathbb{H}\simeq\mathbb{U}$ and (2.24), we get for the second variation of Q,

$$\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0}Q(\mu,\varepsilon\nu)(z)\\
=&\frac{24}{\pi^3}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\rho(z)^{-1}\mu(w)\overline{\nu(u)}\nu(v)d^2vd^2ud^2w}{(w-v)^2(v-\bar{z})^2(w-\bar{u})^2(\bar{u}-\bar{z})^2}\\
+&\frac{24}{\pi^3}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\rho(z)^{-1}\mu(w)\overline{\nu(u)}\nu(v)d^2vd^2ud^2w}{(w-\bar{z})^2(w-v)^2(v-\bar{u})^2(\bar{u}-\bar{z})^2}\\
+&\frac{24}{\pi^3}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\rho(z)^{-1}\mu(w)\overline{\nu(u)}\nu(v)d^2vd^2ud^2w}{(w-\bar{z})^2(w-\bar{u})^2(v-\bar{u})^2(v-\bar{z})^2}\\
-&\mu(z)\rho^{-1}(z)\;\frac{\partial^2}{\partial\varepsilon\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0}\rho^{\varepsilon\nu}(z)=I_1(z)+I_2(z)+I_3(z)+I_4(z).
\end{aligned}$$

Using Proposition 6.3 and property **RK3**, we immediately get that $I_4\in L^2(\mathbb{U},\rho(z)d^2z)$. Now using the Cauchy-Schwarz inequality, the identity

$$\iint\limits_{\mathbb{U}}\frac{d^2w}{|w-\bar{z}|^4}=\frac{\pi}{4}\rho(z),$$

and the property that the Hilbert transform is an isometry on $L^2(\mathbb{C}, d^2z)$, we get

$$\|I_1\|_2^2 = \frac{24^2}{\pi^6} \iint_{\mathbb{U}} \left| \iint_{\mathbb{U}} \iint_{\mathbb{U}} \iint_{\mathbb{U}} \frac{\mu(w)\overline{\nu(u)}\nu(v) d^2v d^2u d^2w}{(w-v)^2(v-\bar{z})^2(w-\bar{u})^2(\bar{u}-\bar{z})^2} \right|^2 \rho(z)^{-1} d^2z$$

$$\leq \frac{24^2}{\pi^6} \iint_{\mathbb{U}} \left(\iint_{\mathbb{U}} \frac{\rho(z)^{-1}|\nu(v)|^2 d^2v}{|v-\bar{z}|^4} \right)$$

$$\times \left(\iint_{\mathbb{U}} \left| \iint_{\mathbb{U}} \iint_{\mathbb{U}} \frac{\mu(w)\overline{\nu(u)} d^2u d^2w}{(w-v)^2(w-\bar{u})^2(\bar{u}-\bar{z})^2} \right|^2 d^2v \right) d^2z$$

$$\leq \frac{12^2\|\nu\|_\infty^2}{\pi^5} \iint_{\mathbb{U}} \iint_{\mathbb{U}} \left| \iint_{\mathbb{U}} \iint_{\mathbb{U}} \frac{\mu(w)\overline{\nu(u)} d^2u d^2w}{(w-v)^2(w-\bar{u})^2(\bar{u}-\bar{z})^2} \right|^2 d^2v d^2z$$

$$\leq \frac{12^2\|\nu\|_\infty^2}{\pi^3} \iint_{\mathbb{U}} \iint_{\mathbb{U}} |\mu(v)|^2 \left| \iint_{\mathbb{U}} \frac{\nu(u) d^2u}{(u-\bar{v})^2(u-z)^2} \right|^2 d^2z d^2v \leq 36\|\nu\|_\infty^4 \|\mu\|_2^2.$$

Similarly, denoting by $\overline{\mathbb{U}}$ the lower half-plane,

$$\|I_3\|_2^2 = \frac{24^2}{\pi^6} \iint_{\mathbb{U}} \left| \iint_{\mathbb{U}} \iint_{\mathbb{U}} \iint_{\mathbb{U}} \frac{\mu(w)\overline{\nu(u)}\nu(v) d^2v d^2u d^2w}{(w-\bar{z})^2(w-\bar{u})^2(v-\bar{u})^2(v-\bar{z})^2} \right|^2 \rho(z)^{-1} d^2z$$

$$\leq \frac{12^2\|\nu\|_\infty^2}{\pi^5} \iint_{\mathbb{U}} \iint_{\mathbb{U}} \left| \iint_{\mathbb{U}} \iint_{\mathbb{U}} \frac{\mu(w)\overline{\nu(u)} d^2u d^2w}{(w-\bar{z})^2(w-\bar{u})^2(v-\bar{u})^2} \right|^2 d^2v d^2z$$

$$= \frac{12^2\|\nu\|_\infty^2}{\pi^5} \iint_{\mathbb{U}} \iint_{\overline{\mathbb{U}}} \left| \iint_{\mathbb{U}} \iint_{\mathbb{U}} \frac{\mu(w)\overline{\nu(u)} d^2u d^2w}{(w-\bar{z})^2(w-\bar{u})^2(\bar{v}-\bar{u})^2} \right|^2 d^2v d^2z$$

$$\leq \frac{12^2\|\nu\|_\infty^2}{\pi^3} \iint_{\mathbb{U}} \iint_{\mathbb{U}} |\nu(v)|^2 \left| \iint_{\mathbb{U}} \frac{\mu(w) d^2w}{(w-\bar{z})^2(w-\bar{v})^2} \right|^2 d^2v d^2z$$

$$\leq \frac{12^2\|\nu\|_\infty^4}{\pi} \iint_{\mathbb{U}} \iint_{\mathbb{U}} \frac{|\mu(v)|^2}{|v-\bar{z}|^4} d^2v d^2z = 36\|\nu\|_\infty^4 \|\mu\|_2^2.$$

For the term I_2 we use the same identity as in the proof of Proposition 6.3[8]

$$\begin{aligned}\frac{1}{(v-\bar{u})^2(\bar{u}-\bar{z})^2} =&\frac{(v-z)^2}{(v-\bar{z})^2(z-\bar{u})^2(v-\bar{u})^2}+\frac{2(v-z)^2(z-\bar{z})}{(v-\bar{z})^3(v-\bar{u})(z-\bar{u})^3}\\ &+\frac{(z-\bar{z})^2}{(v-\bar{z})^2(z-\bar{u})^2(\bar{u}-\bar{z})^2}+\frac{2(z-\bar{z})^2(v-z)}{(v-\bar{z})^3(\bar{u}-\bar{z})(\bar{u}-z)^3}.\end{aligned}$$

As in the proof of Proposition 6.3, the last two terms do not contribute to I_2, and we obtain

$$\begin{aligned}I_2(z) =& \frac{24}{\pi^3}\rho^{-1}(z)\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\mu(w)\nu(v)\overline{\nu(u)}(v-z)^2d^2vd^2ud^2w}{(w-\bar{z})^2(w-v)^2(v-\bar{z})^2(z-\bar{u})^2(v-\bar{u})^2}\\ &+\frac{48}{\pi^3}\rho^{-1}(z)\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\mu(w)\nu(v)\overline{\nu(u)}(v-z)^2(z-\bar{z})d^2vd^2ud^2w}{(w-\bar{z})^2(w-v)^2(v-\bar{z})^3(z-\bar{u})^3(v-\bar{u})}\\ &= I_5(z)+I_6(z).\end{aligned}$$

The L^2-norm of I_5 is estimated exactly as before and we get $\|I_5\|_2^2 \leq 36\|\nu\|_\infty^4\|\mu\|_2^2$. Finally,

$$\begin{aligned}\|I_6\|_2^2 =&\frac{48^2}{\pi^6}\iint\limits_{\mathbb{U}}\left|\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\mu(w)\nu(v)\overline{\nu(u)}(v-z)^2(z-\bar{z})d^2vd^2ud^2w}{(w-\bar{z})^2(w-v)^2(v-\bar{z})^3(z-\bar{u})^3(v-\bar{u})}\right|^2\rho(z)^{-1}d^2z\\ \leq&\frac{24^2\|\nu\|_\infty^2}{\pi^5}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{|z-\bar{z}|^2}{|u-\bar{z}|^2}\left|\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{\mu(w)\nu(v)(v-z)^2d^2vd^2w}{(w-\bar{z})^2(w-v)^2(v-\bar{z})^3(v-\bar{u})}\right|^2d^2ud^2z\\ \leq&\frac{24^2\|\nu\|_\infty^2}{\pi^5}\iint\limits_{\mathbb{U}}\left(\iint\limits_{\mathbb{U}}\left|\iint\limits_{\mathbb{U}}\frac{\mu(w)d^2w}{(w-\bar{z})^2(w-v)^2}\right|^2d^2v\right)\\ &\left(\iint\limits_{\mathbb{U}}\frac{|z-\bar{z}|^2}{|u-\bar{z}|^2}\iint\limits_{\mathbb{U}}\frac{|\nu(v)|^2|v-z|^4}{|v-\bar{z}|^6|v-\bar{u}|^2}d^2vd^2u\right)d^2z.\end{aligned}$$

Making a change of variables $\frac{v-z}{v-\bar{z}}\mapsto v$ and $\frac{u-z}{u-\bar{z}}\mapsto u$, we obtain

$$\begin{aligned}&\iint\limits_{\mathbb{U}}\frac{|z-\bar{z}|^2}{|u-\bar{z}|^2}\iint\limits_{\mathbb{U}}\frac{|\nu(v)|^2|v-z|^4}{|v-\bar{z}|^6|v-\bar{u}|^2}d^2vd^2u \quad\leq \|\nu\|_\infty^2\iint\limits_{\mathbb{D}}\iint\limits_{\mathbb{D}}\frac{|v|^4}{|1-v\bar{u}|^2}d^2vd^2u\\ &=\frac{3\pi^2}{4}\|\nu\|_\infty^2.\end{aligned}$$

[8]We could estimate I_2 in the same way as I_1 If $\nu \in H^{-1,1}(\mathbb{U})$. However, for Theorem 8.5 we only have $\nu \in \Omega^{-1,1}(\mathbb{U})$.

Hence

$$\|I_6\|_2^2 \leq \frac{3\cdot 12^2\|\nu\|_\infty^4}{\pi}\iint\limits_{\mathbb{U}}\iint\limits_{\mathbb{U}}\frac{|\mu(v)|^2}{|v-\bar{z}|^4}d^2v d^2z = 108\|\nu\|_\infty^4\|\mu\|_2^2.$$

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